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HOMOTOPY SPECTRAL SEQUENCES AND OBSTRUCTIONS

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ABSTRACT

For a pointed cosimplicial space $X^\bullet$, the author and Kan developed a spectral sequence abutting to the homotopy of the total space Tot $X^\bullet$. In this paper, $X^\bullet$ is allowed to be unpointed and the spectral sequence is extended to include terms of negative total dimension. Improved convergence results are obtained, and a very general homotopy obstruction theory is developed with higher order obstructions belonging to spectral sequence terms. This applies, for example, to the classical homotopy spectral sequence and obstruction theory for an unpointed mapping space, as well as to the corresponding unstable Adams spectral sequence and associated obstruction theory, which are presented here.

§1. Introduction

Recall from [8] that Tot $X^\bullet$ is the mapping space Map($\Delta^\bullet, X^\bullet$) where $\Delta^\bullet$ is the cosimplicial space of standard simplices Δ^m for $m \geqq 0$. Assuming that $X^\bullet$ is made fibrant, Tot $X^\bullet$ is also the inverse limit of the tower $\{\mathrm{Tot}_s X^\bullet\}$ of fibrations with $\mathrm{Tot}_s X^\bullet = \mathrm{Map}(Sk_s\Delta^\bullet, X^\bullet)$ for $s \geqq 0$ where Sk_s is the s-skeleton functor. The spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$ of [8], abutting to $\{\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)\}$ for a vertex $b \in \mathrm{Tot}\, X^\bullet$, was constructed as the homotopy spectral sequence of the tower $\{(\mathrm{Tot}_s X^\bullet, b)\}$ and has $E_2^{s,t}(X^\bullet, b) = \pi^s\pi_t(X^\bullet, b)$ for $t \geqq s$. It was "fringed" in dimension zero, since there were no negative dimensional terms to receive differentials. This construction has long seemed unnecessarily restricted and can obviously be extended at the E_2-level. Acting in the spirit of [9], we obtain our present version of the spectral sequence by constructing an array of differential relations on the nor-

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malized homotopy of $X^\bullet$, using universal cosimplicial models when needed. For any vertex $b \in \mathrm{Tot}_m X^\bullet$ liftable to $\mathrm{Tot}_{2m} X^\bullet$ with $m \geqq 1$, we obtain a truncated spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}_{1 \leq r \leq m+1}$ where $E_r^{s,t}(X^\bullet, b)$ is defined for

$$t \geqq s \geqq 0 \quad \text{or} \quad t - r \geqq \left(\frac{r-2}{r-1}\right)(s - r) \quad \text{with } s \geqq r,$$

and where $E_r^{s,t}(X^\bullet, b)$ depends only on the projection of b to $\mathrm{Tot}_{r-1} X^\bullet$. As usual, $E_{r+1}(X^\bullet, b)$ is determined by the action of d_r on $E_r(X^\bullet, b)$, although certain differentials are relational. For a vertex $b \in \mathrm{Tot}\, X^\bullet$ this produces the desired spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$. For a vertex $b \in \mathrm{Tot}_n X^\bullet$ with $n \geqq 0$, we define natural obstruction elements $\gamma_r(b) \in E_r^{n+1,n}(X^\bullet, b)$ for $1 \leqq r \leqq (n+2)/2$, and show that $\gamma_r(b) = 0$ iff the projection of b to $\mathrm{Tot}_{n-r+1} X^\bullet$ lifts to $\mathrm{Tot}_{n+1} X^\bullet$. Here we use our truncated spectral sequences. We define similar obstructions to lifting paths. Our homotopy spectral sequence and obstruction results have improved versions when certain Whitehead products vanish in $X^\bullet$.

We obtain results on the convergence of $\{E_r^{s,t}(X^\bullet, b)\}$ to $\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)$ for $t - s \geqq 0$, and we derive a comparison theorem showing that a cosimplicial map $f: X^\bullet \to Y^\bullet$ induces an equivalence $\mathrm{Tot}\, X^\bullet \simeq \mathrm{Tot}\, Y^\bullet$ when it induces a suitable E_r-equivalence for some r. We also obtain a natural Hurewicz map from $\{E_r^{s,t}(X^\bullet, b)\}$ to the R-homology spectral sequence $\{E_r^{s,t}(X^\bullet; R)\}$ of [6] for a ring R. In fact, our sign conventions are chosen to permit this. Turning to examples, we first discuss an extended version of the classical homotopy spectral sequence and obstruction theory for a mapping space $\mathrm{Map}(K, L)$ with E_2-term $\{H^s(K; \pi_t L)\}$ and with a Hurewicz map to an Anderson homology spectral sequence. We then discuss the corresponding unstable Adams spectral sequence and obstruction theory with E_2-term $\{\mathrm{Der}^{s,t}_{\mathcal{CA}}(H_* K, H_* L)\}$ and with a Hurewicz map to a homology spectral sequence involving derived Lannes functors. Many other examples can be developed; for instance, our machinery applies to the homotopy inverse limit spectral sequences of [8] and to the unstable Adams spectral sequences of Dwyer–Miller–Neisendorfer [12] and Dror–Zabrodsky [11].

The paper is organized as follows: Section 2 presents our general homotopy spectral sequence; Section 3 explains its agreement with the tower homotopy spectral sequence; Section 4 contains convergence results; Section 5 develops our cosimplicial obstruction theory; Section 6 gives connectivity and comparison results for total spaces; Section 7 extends the classical homotopy spectral sequence and obstruction theory of a mapping space; Section 8 deals with derivations over the Steenrod algebra; Section 9 develops the unstable Adams spectral sequence and obstruction theory of a mapping space; Sections 10–13 contain detailed constructions underlying our results; and Section 14 is an appendix on the homotopy theory of groupoids and cosimplicial groupoids, with a general application to the lifting problem for vertices in a tower of total spaces.

This paper extends joint work with D. M. Kan and constitutes an expanded response to some questions posed by E. Dror-Farjoun, H. Miller, and J. Neisendorfer. We work simplicially and generally follow the terminology of [8], so that "space" means "simplicial set".

§2. The homotopy spectral sequence of a cosimplicial space

After needed preliminaries, we present our general homotopy spectral sequence, postponing the main constructions and proofs to Sections 10–13. Our domains of definition for spectral sequence terms are not best possible but seem most convenient. Throughout this section, $X^\bullet$ will be a fibrant cosimplicial space.

2.1. The homotopy of $X^\bullet$. For $t \geqq 0$, $X^\bullet$ has *unpointed normalized homotopy* $N\pi_t^{\text{free}}X^m$ consisting of all $x \in \pi_t^{\text{free}}X^m = [S^t, X^m]_{\text{free}}$ such that $s^j x \in \pi_t^{\text{free}}X^{m-1}$ is trivial for $0 \leqq j \leqq m$, and for a vertex $v \in X^m$, $X^\bullet$ has *normalized homotopy*

$$N\pi_t(X^m, v) = \bigcap_{j=0}^{m-1} \ker(s^j_* : \pi_t(X^m, v) \to \pi_t(X^{m-1}, s^j v)).$$

For a vertex $b \in \operatorname{Tot}_q X^\bullet$, let b also denote the projected vertex $b_k \in \operatorname{Tot}_k X^\bullet$ for $k \leqq q$ and the vertex $(d^1)^m b_0 \in X^m$ for $m \geqq 0$. Thus $N\pi_t(X^m, b)$ denotes $N\pi_t(X^m, (d^1)^m b_0)$. The homomorphism $(d^1)^m_* : \pi_1(X^0, b) \to \pi_1(X^m, b)$ induces a right action by $\pi_1(X^0, b)$ on $N\pi_t(X^m, b) \subset \pi_t(X^m, b)$ for $t \geqq 1$. Recall that a vertex $b \in \operatorname{Tot}_q X^\bullet$ is a cosimplicial map $b : Sk_q\Delta^\bullet \to X^\bullet$. For $t \geqq 1$ there is a *cosimplicial group* $\pi_t(X^\bullet, b) = \{\pi_t(X^m, b)\}_{m \geqq 0}$ for each vertex $b \in \operatorname{Tot}_1 X^\bullet$ such that, for each $m \geqq 0$, $\pi_1(Sk_1\Delta^m, 0)$ acts trivially on $\pi_t(X^m, b)$ via $b_* : \pi_1(Sk_1\Delta^m, 0) \to \pi_1(X^m, b)$ where $0 \in \Delta^m$ is the initial vertex. This is automatic when $b \in \operatorname{Tot}_1 X^\bullet$ lifts to $\operatorname{Tot}_2 X^\bullet$, or when the spaces X^m have simple components.

2.2. The cohomotopy of a cosimplicial group. A cosimplicial group $G^\bullet$ has normalized groups

$$NG^m = \bigcap_{j=0}^{m-1} \ker(s^j : G^m \to G^{m-1})$$

for $m \geqq 1$. When $G^\bullet$ is abelian there are coboundary homomorphisms

$$\delta = \sum_{j=0}^{m+1} (-1)^j d^j : NG^m \to NG^{m+1}$$

giving *cohomotopy groups* $H^m(NG^\bullet) = \pi^m G^\bullet$. In general, the group NG^0 right-acts on each group NG^m by

$$a \cdot x = [(d^1)^m x]^{-1} a [(d^1)^m x] \qquad \text{for } x \in NG^0 \quad \text{and} \quad a \in NG^m.$$

There is a coboundary function $\delta : NG^0 \to NG^1$ with $\delta x = (d^1 x)^{-1}(d^0 x)$ which is a crossed-homomorphism and determines (see below) an associated right-action $*$ of the group NG^0 on the set NG^1 with $y * x = (d^1 x)^{-1} y (d^0 x)$. There is also a coboundary function $\delta : NG^1 \to NG^2$ with $\delta y = (d^2 y)(d^0 y)(d^1 y)^{-1}$ which satisfies $\delta(y * x) = (\delta y) \cdot x$ for each $y \in NG^1$ and $x \in NG^0$. The *cohomotopy group* $\pi^0 G^\bullet$ is the kernel of $\delta : NG^0 \to NG^1$, and the *pointed cohomotopy set* $\pi^1 G^\bullet$ is the quotient of the kernel of $\delta : NG^1 \to NG^2$ by the $*$-action of NG^0. There is no reasonable coboundary function $\delta : NG^m \to NG^{m+1}$ for $m \geqq 2$, but we shall use the alternating product

$$\delta(x) = (d^0 x)(d^1 x)^{-1}(d^2 x) \cdots (d^{m+1} x)^{\pm 1}$$

when necessary, although $\delta\delta$ need not be trivial. Generalizing $\pi^0 G^\bullet$, we let

$$\pi^0 J^\bullet = \{x \in J^0 \mid d^0 x = d^1 x\}$$

for a cosimplicial set $J^\bullet$. For a group B right-acting on an additive (but possibly non-abelian) group M, a *crossed-homomorphism* $f : B \to M$ is a function with $f(ab) = (fa)b + fb$ for each $a, b \in B$. The *crossed-homomorphic action* of B on the set M is defined by $m * b = mb + fb$ for $m \in M$ and $b \in B$.

2.3. Relations. A *relation* $f : A \rightharpoonup B$ from a set A to a set B is a subset $f \subset A \times B$, with the notation $f(a) = b$ indicating $(a, b) \in f$. We define

$$\text{domain} f = \{a \in A \mid f(a) = b \text{ for some } b \in B\},$$

$$\text{image} f = \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

When A and B are pointed, by $0 \in A$ and $0 \in B$, we define

$$\text{kernel} f = \{a \in A \mid f(a) = 0\},$$

$$\text{indeterminacy} f = \{b \in B \mid f(0) = b\},$$

and we call f *pointed* when $f(0) = 0$. We can now introduce

2.4. The spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$. For $r \geqq 1$, let $b \in \mathrm{Tot}_{r-1} X^\bullet$ be a vertex liftable to $\mathrm{Tot}_{2r-2} X^\bullet$. When $r = 1$ and $s \geqq 0$, $E_1^{s,t}(X^\bullet, b) = N\pi_t(X^s, b)$ as a pointed set for $t = 0$, as a group with right $E_1^{0,1}(X^\bullet, b)$-action for $t = 1$, and as an abelian group with right $E_1^{0,1}(X^\bullet, b)$ action for $t \geqq 2$. When $r = 2$, $E_2^{s,t}(X^\bullet, b) = \pi^s\pi_t(X^\bullet, b)$ as a pointed set for $(s, t) = (0, 0)$ or $(s, t) = (1, 1)$, as a group for $(s, t) = (0, 1)$, and as an abelian group with right $E_2^{0,1}(X^\bullet, b)$-action for $t \geqq 2$ with $s \geqq 0$. When $r \geqq 2$, $E_r^{s,t}(X^\bullet, b)$ is defined for

$$t \geqq s \geqq 0 \quad \text{or} \quad t - r \geqq \left(\frac{r-2}{r-1}\right)(s - r) \quad \text{with } s \geqq r:$$

it is a pointed set for $0 \leqq s = t \leqq r - 1$, a group for $(s, t) = (0, 1)$, and an abelian group with right $E_r^{0,1}(X^\bullet, b)$-action otherwise. The terms $E_r^{s,t}(X^\bullet, b)$ are natural in $b \in \mathrm{Tot}_{r-1} X^\bullet$ and $X^\bullet$, i.e. for path classes in $\mathrm{Tot}_{r-1} X^\bullet$ and cosimplicial maps. Next, assume that $E_{r+1}^{s,t}(X^\bullet, b)$ is defined; thus, for $r \geqq 1$, let $b \in \mathrm{Tot}_r X^\bullet$ be a vertex liftable to $\mathrm{Tot}_{2r} X^\bullet$ and suppose that

$$t \geqq s \geqq 0 \quad \text{or} \quad t - r - 1 \geqq \left(\frac{r-1}{r}\right)(s - r - 1) \quad \text{with } s \geqq r + 1.$$

Then there is a differential d_r going out of $E_r^{s,t}(X^\bullet, b)$ and consisting of: a pointed relation $d_r : E_r^{0,0}(X^\bullet, b) \rightharpoonup N\pi_{r-1}^{\text{free}} X^r$ with domain $E_r^{0,0}(X^\bullet, b)$; a pointed relation

$$d_r : E_r^{t,t}(X^\bullet, b) \rightharpoonup E_t^{t+r,t+r-1}(X^\bullet, b)/E_t^{0,1}(X^\bullet, b)$$

for $1 \leqq t \leqq r - 1$ with domain $E_r^{t,t}(X^\bullet, b)$ where "/" forms the orbit set; a crossed-homomorphism $d_r : E_r^{0,1}(X^\bullet, b) \to E_r^{r,r}(X^\bullet, b)$; a pointed function $d_r : E_r^{r,r}(X^\bullet, b) \to E_r^{2r,2r-1}(X^\bullet, b)$ such that

$$d_r(y * x) = (d_r y)x \qquad \text{for } x \in E_r^{0,1}(X^\bullet, b) \quad \text{and} \quad y \in E_r^{r,r}(X^\bullet, b)$$

using the crossed homomorphism action $y * x = yx + d_r x$; and an $E_r^{0,1}(X^\bullet, b)$-equivariant homomorphism $d_r : E_r^{s,t}(X^\bullet, b) \to E_r^{s+r,t+r-1}(X^\bullet, b)$ otherwise. The differential d_r is natural in $b \in \mathrm{Tot}_r X^\bullet$ and $X^\bullet$. It satisfies $d_r d_r = 0$ whenever the composition is defined, and $E_{r+1}^{s,t}(X^\bullet, b)$ is given by: the set of all $x \in E_r^{0,0}(X^\bullet, b)$ with $d_r x = y$ for some trivial $y \in N\pi_{r-1}^{\text{free}} X^r$ when $(s, t) = (0, 0)$; the kernel of d_r on $E_r^{s,t}(X^\bullet, b)$ when $s < r$ and $(s, t) \neq (0, 0)$; the orbit set of the kernel of d_r on $E_r^{r,r}(X^\bullet, b)$ under the crossed-homomorphism action by $E_r^{0,1}(X^\bullet, b)$ when $(s, t) = (r, r)$; and the ordinary homology with respect to d_r at $E_r^{s,t}(X^\bullet, b)$ otherwise. When $r = 1$, the differential

$$d_1 : E_1^{s,t}(X^\bullet, b) \to E_1^{s+1,t}(X^\bullet, b)$$

equals $(-1)^{t-s-1}\delta : N\pi_t(X^s, b) \to N\pi_t(X^{s+1}b)$ in the cosimplicial group $\pi_t(X^\bullet, b)$ for $t \geqq 1$, and the differential

$$d_1 : E_1^{0,0}(X^\bullet, b) \to N\pi_0^{\text{free}} X^1$$

equals $(d^0, d^1)_* : \pi_0 X^0 \to N\pi_0^{\text{free}} X^1$. We thereby recover the fact that $E_2^{s,t}(X^\bullet, b) = \pi^s \pi_t(X^\bullet, b)$.

For a vertex $b \in \text{Tot}\, X^\bullet$, we now have a spectral sequence $\{E_r(X^\bullet, b)\}_{1 \leq r < \infty}$ which, we shall see, generalizes and extends the spectral sequence of [8] and [9]. Likewise, for a vertex $b \in \text{Tot}_m X^\bullet$ liftable to $\text{Tot}_{2m} X^\bullet$ with $0 \leqq m < \infty$, we have a truncated spectral sequence $\{E_r(X^\bullet, b)\}_{1 \leq r \leq m+1}$. In general, $E_r(X^\bullet, b)$ and d_{r-1} depend only on the projected vertex $b \in \text{Tot}_{r-1} X^\bullet$; however, when Whitehead products vanish in $X^\bullet$, they will depend only on $b \in \text{Tot}_{r-2} X^\bullet$ and will be more widely defined as follows.

2.5. On $\{E_r^{s,t}(X^\bullet, b\}$ when Whitehead products vanish. Suppose that all Whitehead products vanish in the spaces $X^\bullet$ for $s \geqq 0$. This is automatic when $X^\bullet$ is "grouplike" as in [8, p. 275]. For $r \geqq 2$, let $b \in \text{Tot}_{r-2} X^\bullet$ be a vertex liftable to $\text{Tot}_{2r-3} X^\bullet$. When $r = 2$,

$$E_1^{s,t}(X^\bullet, b) = N\pi_t(X^s, b) \quad \text{and} \quad E_2^{s,t}(X^\bullet, b) = \pi^s \pi_t(X^\bullet, b)$$

as pointed sets for $(s, t) = (0, 0)$ and as abelian groups for $t = 1$ with $s \geqq 0$. When $r \geqq 2$, $E_r^{s,t}(X^\bullet, b)$ is defined for

$$t \geqq s \geqq 0 \quad \text{or} \quad t - r + 1 \geqq \left(\frac{r-2}{r-1}\right)(s - r + 1) \quad \text{with } s \geqq r:$$

it is a pointed set for $0 \leqq s = t \leqq r - 2$ and an abelian group otherwise. The terms $E_r^{s,t}(X^\bullet, b)$ are natural in $b \in \text{Tot}_{r-2} X^\bullet$ and $X^\bullet$, i.e for path classes in $\text{Tot}_{r-2} X^\bullet$ and cosimplicial maps. Next assume that $E_{r+1}^{s,t}(X^\bullet, b)$ is defined; thus, for $r \geqq 1$, let $b \in \text{Tot}_{r-1} X^\bullet$ be a vertex liftable to $\text{Tot}_{2r-1} X^\bullet$, and suppose that

$$t \geqq s \geqq 0 \quad \text{or} \quad t - r \geqq \left(\frac{r-1}{r}\right)(s-r) \quad \text{with } s \geqq r+1.$$

Then there is a differential d_r going out of $E_r^{s,t}(X^\bullet, b)$ and consisting of: a pointed relation $d_r : E_r^{0,0}(X^\bullet, b) \rightharpoonup N\pi_{r-1}^{\text{free}} X^r$ with domain $E_r^{0,0}(X^\bullet, b)$; a pointed relation

$$d_r : E_r^{t,t}(X^\bullet, b) \rightharpoonup E_{t+1}^{t+r,t+r-1}(X^\bullet, b) \qquad \text{for } 1 \leqq t \leqq r-2$$

with domain $E_r^{t,t}(X^\bullet, b)$; a pointed function

$$d_r : E_r^{r-1,r-1}(X^\bullet, b) \to E_r^{2r-1,2r-2}(X^\bullet, b) \quad \text{when } r \geqq 2;$$

and a homomorphism $d_r : E_r^{s,t}(X^\bullet, b) \to E_r^{s+r,t+r-1}(X^\bullet, b)$ otherwise. The differential d_r is natural in $b \in \text{Tot}_{r-1} X^\bullet$ and $X^\bullet$. It satisfies $d_r d_r = 0$ whenever the composition is defined, and $E_{r+1}^{s,t}(X^\bullet, b)$ is given by the homology with respect to d_r at $E_r^{s,t}(X^\bullet, b)$ as in 2.4.

Consequently, when $X^\bullet$ has vanishing Whitehead products, we have a spectral sequence $\{E_r(X^\bullet, b)\}_{1 \leq r < \infty}$ for $b \in \text{Tot}\, X^\bullet$ and a truncated spectral sequence $\{E_r(X^\bullet, b)\}_{1 \leq r \leq m+2}$ for $b \in \text{Tot}_m X^\bullet$ liftable to $\text{Tot}_{2m+1} X^\bullet$ with $0 \leqq m < \infty$. These extend the spectral sequences of 2.4.

2.6. Technical refinements. The notation $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ will indicate $[\pi_t(X^s, v), \pi_j(X^s, v)] = 0$ in $\pi_{t+j-1}(X^s, v)$ for each $j \geqq 1$, $s \geqq 0$, and $v \in X^s$. For $m \geqq 0$, let $b \in \text{Tot}_m X^\bullet$ be a vertex satisfying either of the conditions: (i) b is liftable to $\text{Tot}_{2m+1} X^\bullet$ and $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq 2m+1$; or (ii) b is liftable to $\text{Tot}_{2m+2} X^\bullet$ and $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq m+1$. Then there is a truncated spectral sequence $\{E_r(X^\bullet, b)\}_{1 \leq r \leq m+2}$ exactly as in 2.5. Next, for $m \geqq 2$, let $b \in \text{Tot}_m X^\bullet$ be a vertex liftable to $\text{Tot}_{2m} X^\bullet$. If $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for some t with $1 \leqq t \leqq m-1$, then there is a natural pointed relation

$$d_m : E_m^{t,t}(X^\bullet, b) \rightharpoonup E_{t+1}^{t+m,t+m-1}(X^\bullet, b)$$

with domain $E_m^{t,t}(X^\bullet, b)$ and kernel $E_{m+1}^{t,t}(X^\bullet, b)$; this d_m is a function when $t = m-1$.

2.7. Hurewicz maps of spectral sequences. For a pointed space (Y, y_0) and ring R with identity, let $h : \pi_t(Y, y_0) \to H_t(Y; R)$ be the Hurewicz map given by the composite of the forgetful map $\pi_t(Y, y_0) \to \pi_t^{\text{free}} Y$ with the unpointed Hurewicz map $h : \pi_t^{\text{free}} Y \to H_t(Y; R)$. Thus h is a homomorphism for $t \geqq 1$ and $h[y] = [y - y_0]$ for $t = 0$. This will induce a Hurewicz map h from the (possibly truncated) homotopy spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$ of 2.4–2.6 to the homology spectral sequence $\{E_r^{s,t}(X^\bullet, R)\}$ of [6] and 10.8. The spectral sequence map h will begin with the obvious maps

$$h : N\pi_t(X^s, b) \to NH_t(X^s; R)$$

for $r = 1$ and

$$h : \pi^s\pi_t(X^s, b) \to \pi^s H_t(X^s; R)$$

for $r = 2$, and it will respect all differentials, including the relational ones. For a vertex $b \in \operatorname{Tot} X^\bullet$, the Hurewicz map $h : \{E_r^{s,t}(X^\bullet, b)\} \to \{E_r^{s,t}(X^\bullet; R)\}$ will abut to the Hurewicz map $h : \pi_{t-s}(\operatorname{Tot} X^\bullet, b) \to H_{t-s}(\operatorname{Tot} X^\bullet; R)$.

§3. Agreement with the tower spectral sequence

For a fibrant cosimplicial space $X^\bullet$, we now interpret the terms $E_r^{s,t}(X^\bullet, b)$ for $t \geqq s$ as derived homotopy groups of fibers in $\{\operatorname{Tot}_s X^\bullet\}$ when b is sufficiently liftable. We thus see that our present spectral sequence extends that of [8]. As in [8], but for non-pointed $X^\bullet$, we use

3.1. The derived homotopy exact sequence. For $r \geqq 1$ and $s \geqq 0$, let $b \in \operatorname{Tot}_{s+r-1} X^\bullet$ be a vertex. Then for $i \geqq 0$, let $\pi_i(\operatorname{Tot}_s X^\bullet, b)^{(r-1)}$ denote the image of

$$\pi_i(\operatorname{Tot}_{s+r-1} X^\bullet, b) \to \pi_i(\operatorname{Tot}_s X^\bullet, b).$$

Consider the fiber $\operatorname{Fib}_s(X^\bullet, b)$ of $\operatorname{Tot}_s X^\bullet \to \operatorname{Tot}_{s-1} X^\bullet$ at the projected vertex $b \in \operatorname{Tot}_s X^\bullet$. Let $C_i \subset \pi_i \operatorname{Fib}_s(X^\bullet, b)$ be the counterimage of $\pi_i(\operatorname{Tot}_s X^\bullet, b)^{(r-1)}$ under $\pi_i \operatorname{Fib}_s(X^\bullet, b) \to \pi_i(\operatorname{Tot}_s X^\bullet, b)$, and let $K_{i+1} \subset \pi_{i+1}(\operatorname{Tot}_{s-1} X^\bullet, b)$ be the kernel of

$$\pi_{i+1}(\operatorname{Tot}_{s-1} X^\bullet, b) \to \pi_{i+1}(\operatorname{Tot}_{s-r} X^\bullet, b).$$

Then form the group

$$\pi_i \operatorname{Fib}_s(X^\bullet, b)^{(r-1)} = C_i/\partial_* K_{i+1} \qquad \text{for } i \geqq 1$$

and the orbit set $\pi_0 \operatorname{Fib}_s(X^\bullet, b)^{(r-1)}$ of C_0 under the fibration right-action by K_1 when $i = 0$. There is now a *derived homotopy exact sequence*

$$\cdots \longrightarrow \pi_{i+1}(\operatorname{Tot}_{s-r+1} X^\bullet, b)^{(r-1)} \to \pi_{i+1}(\operatorname{Tot}_{s-r} X^\bullet, b)^{(r-1)}$$

$$\xrightarrow{\partial_*} \pi_i \operatorname{Fib}_s(X^\bullet, b)^{(r-1)} \to \pi_i(\operatorname{Tot}_s X^\bullet, b)^{(r-1)} \to \pi_i(\operatorname{Tot}_{s-1} X^\bullet, b)^{(r-1)}$$

which is natural in $X^\bullet$ and $b \in \operatorname{Tot}_{s+r-1} X^\bullet$, i.e. for cosimplicial maps and path classes in $\mathrm{Tot}_{s+r-1} X^\bullet$. All of its maps are group homomorphisms except for the last three when $i=0$. However, there is a fibration right-action $*$ by $\pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$ on the set $\pi_0 \mathrm{Fib}_s(X^\bullet, b)^{(r-1)}$, such that $\partial_*(g) = 0 * g$ for each $g \in \pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$ and such that elements of $\pi_0(\mathrm{Fib}_s(X^\bullet, b)^{(r-1)}$ are in the same orbit iff they have the same image in $\pi_0(\mathrm{Tot}_s X^\bullet, b)^{(r-1)}$. The formula $\partial_*(g) = 0 * g$ implies that elements of $\pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$ are in the same right coset of $\ker \partial_*$ iff they have the same image in $\pi_0 \mathrm{Fib}_s(X^\bullet, b)^{(r-1)}$. Letting d_r denote the composite of

$$\pi_{i+1}(\mathrm{Fib}_{s-r} X^\bullet, b)^{(r-1)} \to \pi_{i+1}(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)} \xrightarrow{\partial_*} \pi_i \mathrm{Fib}_s(X^\bullet, b)^{(r-1)},$$

when $s \geqq r$, we recover the homotopy spectral sequence of [8, p. 281] with E_r-term $\{\pi_i \mathrm{Fib}_s(X^\bullet, b)^{(r-1)}\}$ when $b \in \mathrm{Tot}\, X^\bullet$.

3.2. Replacing $\pi_i \mathrm{Fib}_s(X^\bullet, b)$ by $N\pi_{i+s}(X^s, b)$. Let $b \in \mathrm{Tot}_s X^\bullet$ be a vertex for $s \geqq 0$. By 10.2, there is a natural isomorphism

$$\Phi : \pi_i \mathrm{Fib}_s(X^\bullet, b) \approx N\pi_{i+s}(X^s, b)$$

of groups for $i \geqq 1$ and of pointed sets for $i = 0$. For $s \geqq k \geqq 0$, $m \geqq 0$, and $t \geqq 1$, the *fundamental right-action* of $\pi_1(\mathrm{Tot}_k X^\bullet, b)$ on the group $N\pi_t(X^m, b)$ is defined via

$$\pi_1(\mathrm{Tot}_k X^\bullet, b) \to \pi_1(\mathrm{Tot}_0 X^\bullet, b) = \pi_1(X^0, b)$$

from the right-action of $\pi_1(X^0, b)$ on $\pi_t(X^m, b)$ in 2.1. By the naturality of Φ, the fundamental action of $\pi_1(\mathrm{Tot}_s X^\bullet, b)$ on $\pi_i \mathrm{Fib}_s(X^\bullet, b) \approx N\pi_{i+s}(X^\bullet, b)$ agrees with the fibration action. By 10.5, for $s \geqq 1$ the fibration boundary

$$\partial_* : \pi_1(\mathrm{Tot}_{s-1} X^\bullet, b) \to \pi_0 \mathrm{Fib}_s(X^\bullet, b) \approx N\pi_s(X^s, b)$$

is a crossed-homomorphim with respect to the fundamental action of $\pi_1(\mathrm{Tot}_{s-1} X^\bullet, b)$, and the crossed-homomorphism action (2.2) agrees with the fibration action of $\pi_1(\mathrm{Tot}_{s-1} X^\bullet, b)$ on $\pi_0 \mathrm{Fib}_s(X^\bullet, b) \approx N\pi_s(X^s, b)$.

3.3. Replacing $\pi_i \mathrm{Fib}_s(X^\bullet, b)^{(r-1)}$ by $E_r^{s,s+i}(X^\bullet, b)$. For $r \geqq 1$ and $s \geqq 0$, let $b \in \mathrm{Tot}_{s+r-1} X^\bullet$ be a vertex liftable to $\mathrm{Tot}_{2r-2} X^\bullet$ (or to $\mathrm{Tot}_{2r-3} X^\bullet$ when $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq 2r-3$). Then by 11.5, the above Φ induces a bijection

$$\Phi : \pi_i \mathrm{Fib}_s(X^\bullet, b)^{(r-1)} \approx E_r^{s,s+i}(X^\bullet, b) \qquad \text{for } i \geqq 0$$

which is a group isomorphism for $i \geqq 1$ and is natural in $X^\bullet$ and b. Using Φ, the *derived homotopy exact sequence* of 3.1 becomes

$$\cdots \longrightarrow \pi_{i+1}(\mathrm{Tot}_{s-r+1} X^\bullet, b)^{(r-1)} \to \pi_{i+1}(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$$

$$\xrightarrow{\partial_*} E_r^{s,s+i}(X^\bullet, b) \to \pi_i(\mathrm{Tot}_s X^\bullet, b)^{(r-1)} \to \pi_i(\mathrm{Tot}_{s-1} X^\bullet, b)^{(r-1)}.$$

Thus, when $r > s \geqq 0$ *and* $i \geqq 0$, $E_r^{s,s+i}(X^\bullet, b)$ *is the kernel of* $\pi_i(\mathrm{Tot}_s X^\bullet, b)^{(r-1)} \to \pi_i(\mathrm{Tot}_{s-1} X^\bullet, b)^{(r-1)}$. By 3.2 when $r \leqq s$,

$$\partial_* : \pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)} \to E_r^{s,s}(X^\bullet, b)$$

is a crossed-homomorphism with respect to the fundamental action of $\pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$ on the group $E_r^{s,s}(X^\bullet, b)$, and the crossed-homomorphism action (2.2) agrees with the fibration action (3.1) of $\pi_1(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)}$ on $E_r^{s,s}(X^\bullet, b)$. Finally, when $s \geqq r$, the composition of

$$E_r^{s-r,s-r+i+1}(X^\bullet, b) \to \pi_{i+1}(\mathrm{Tot}_{s-r} X^\bullet, b)^{(r-1)} \to E_r^{s,s+i}(X^\bullet, b)$$

equals the differential d_r of 2.4. Thus we have

3.4. Agreement of spectral sequences. For any $b \in \mathrm{Tot}\, X^\bullet$, the homotopy spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$ of 2.4 extends the tower spectral sequence of [8, p. 281].

§4. Convergence of the homotopy spectral sequence

For a fibrant cosimplicial space $X^\bullet$ and vertex $b \in \mathrm{Tot}\, X^\bullet$, we now show that $\{E_r^{s,t}(X^\bullet, b)\}$ converges to $\{\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)\}$ under suitable conditions. The problem of initially finding a vertex $b \in \mathrm{Tot}\, X^\bullet$ will be discussed in Sections 5 and 6. First, we need

4.1. Infinitely derived towers of sets. For a tower $\{T_s\}_{s \in Z}$ of sets T_s and functions $T_s \to T_{s-1}$, let $T_s^{(r)}$ denote the image of $T_{s+r} \to T_s$ for $0 \leqq r < \infty$; let $T_s^{(\infty)} = \bigcap_r T_s^{(r)}$; and let $T_s^{(\infty+)}$ denote the image of the projection $\lim_s T_s \to T_s$. Also, for an element $v \in T_{s-1}$ and any r, let $T_s(v)^{(r)}$ denote the set of all $x \in T_s^{(r)}$ projecting to v. Clearly $T_s^{(\infty+)} \subset T_s^{(\infty)}$ and $T_s(v)^{(\infty+)} \subset T_s(v)^{(\infty)}$, where the inclusions may be proper since an element of T_s may be "arbitrarily highly liftable" yet not "consistently infinitely liftable." The condition $T_s^{(\infty+)} = T_s^{(\infty)}$ holds for each s if and only if $T_s(v)^{(\infty+)} = T_s(v)^{(\infty)}$ holds for each s and $v \in T_{s-1}$. These equivalent conditions hold whenever: (i) $\{T_s\}$ can be topologized as a tower of compact Hausdorff spaces and continuous maps; or (ii) the descending sequence $\{T_s(v)^{(r)}\}_{0\le r<\infty}$ is eventually constant for each s and $v\in T_{s-1}$. This follows in case (i) since the inverse limit of a tower of nonempty compact Hausdorff spaces is nonempty by a Tychonoff argument. Better results hold when $\{T_s\}$ is a tower of (possibly nonabelian) groups and homomorphisms with $T_s=0$ for $s<0$. Then, the condition $T_s^{(\infty+)}=T_s^{(\infty)}$ holds for each s if and only if $T_s(0)^{(\infty+)}=T_s(0)^{(\infty)}$ holds for each s. Also then, by the proof of the complete convergence lemma of [8, p. 263], the combined conditions $T_s^{(\infty+)}=T_s^{(\infty)}$ and $\lim^1_m T_m=0$ hold for each s if and only if $\lim^1_r T_s(0)^{(r)}=0$ holds for each s. Moreover, these combined conditions hold whenever: (i) $\{T_s\}$ can be topologized as a tower of compact Hausdorff groups (or linearly compact $\mathscr{C}$-groups [15]) and continuous homomorphisms; or (ii) the descending sequence $\{T_s(0)^{(r)}\}$ is eventually constant for each s. This follows in case (i) by [15, 3.1 and 3.2].

4.2. Complete convergence of $\{E_r^{s,t}(X^\bullet, b)\}$. For a vertex $b\in\operatorname{Tot} X^\bullet$, consider the homotopy spectral sequence $\{E_r^{s,t}(X^\bullet, b)\}$ and let

$$E_\infty^{s,t}(X^\bullet, b)=\bigcap_{r>s} E_r^{s,t}(X^\bullet, b)\qquad \text{for } t\geqq s\geqq 0.$$

Then $E_\infty^{s,t}(X^\bullet, b)$ is a pointed set when $t=s$, a group when $(s,t)=(0,1)$, and an abelian group with right-action by $E_\infty^{0,1}(X^\bullet, b)$ otherwise. By 3.3, $E_r^{s,t}(X^\bullet, b)$ is the kernel of

$$\pi_{t-s}(\operatorname{Tot}_s X^\bullet, b)^{(r-1)}\to\pi_{t-s}(\operatorname{Tot}_{s-1} X^\bullet, b)^{(r-1)}\qquad\text{when } t\geqq s\geqq 0 \quad\text{and}\quad r>s,$$

and thus $E_\infty^{s,t}(X^\bullet, b)$ is the kernel of $\pi_{t-s}(\operatorname{Tot}_s X^\bullet, b)^{(\infty)}\to\pi_{t-s}(\operatorname{Tot}_{s-1} X^\bullet, b)^{(\infty)}$. For $t\geqq s\geqq 0$, let $E_{\infty+}^{s,t}(X^\bullet, b)$ denote the kernel of

$$\pi_{t-s}(\operatorname{Tot}_s X^\bullet, b)^{(\infty+)}\to\pi_{t-s}(\operatorname{Tot}_{s-1} X^\bullet, b)^{(\infty+)}$$

as a pointed set when $t=s$, a group when $(s,t)=(0,1)$, and an abelian group with right-action by $E_{\infty+}^{0,1}(X^\bullet, b)=\pi_1(X^0, b)^{(\infty+)}$ otherwise. Then $E_{\infty+}^{s,t}(X^\bullet, b)\subset E_\infty^{s,t}(X^\bullet, b)$ as a subobject for $t\geqq s\geqq 0$. As in [8, p. 254], there is a short exact sequence

$$0\to\lim{}^1_s\,\pi_{i+1}(\operatorname{Tot}_s X^\bullet, b)\to\pi_i(\operatorname{Tot} X^\bullet, b)\to\lim{}_s\,\pi_i(\operatorname{Tot}_s X^\bullet, b)\to 0$$

of a pointed sets for $i=0$ and groups for $i\geqq 1$. Thus $\pi_i(\operatorname{Tot}_s X^\bullet, b)^{(\infty+)}$ is the image of $\pi_i(\operatorname{Tot} X^\bullet, b)\to\pi_i(\operatorname{Tot}_s X^\bullet, b)$, and $\{\pi_i(\operatorname{Tot}_s X^\bullet, b)^{(\infty+)}\}_{s\geqq 0}$ is a surjective tower with

$$\pi_i(\operatorname{Tot} X^\bullet, b)\to\lim{}_s\,\pi_i(\operatorname{Tot}_s X^\bullet, b)^{(\infty+)}$$

surjective for each $i \geqq 0$. Consequently, the elements in the tower kernels $E_{\infty+}^{s,s+i}(X^\bullet, b)$ are all hit by elements of $\pi_t(\mathrm{Tot}\, X^\bullet, b)$, while the remaining elements of $E_\infty^{s,s+i}(X^\bullet, b)$ are not. Given $i \geqq 0$, we say that $\{E_r^{s,t}(X^\bullet, b)\}$ *converges completely to* $\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)$ *for* $t - s = i$ when $E_{\infty+}^{s,s+i}(X^\bullet, b) = E_\infty^{s,s+i}(X^\bullet, b)$ for all $s \geqq 0$ and $\pi_i(\mathrm{Tot}\, X^\bullet, b) \to \lim_s \pi_i(\mathrm{Tot}_s\, X^\bullet, b)$ has trivial kernel. This last condition holds iff $\lim_s^1 \pi_{i+1}(\mathrm{Tot}_s\, X^\bullet, b)$ is trivial or, when $i \geqq 1$, holds iff $\pi_i(\mathrm{Tot}\, X^\bullet, b) \to \lim_s \pi_i(\mathrm{Tot}_s\, X^\bullet, b)$ is iso.

4.3. Convergence results for $\{E_r^{s,t}(X^\bullet, b)\}$. Applying 4.1 to appropriate subtowers of $\{\pi_i(\mathrm{Tot}_s\, X^\bullet, b)\}$, we obtain the following convergence results at a vertex $b \in \mathrm{Tot}\, X^\bullet$. First as in [8, p. 263], for $i \geqq 1$, the combined conditions $E_{\infty+}^{s,s+i}(X^\bullet, b) = E_\infty^{s,s+i}(X^\bullet, b)$ and $\lim_m^1 \pi_i(\mathrm{Tot}_m\, X^\bullet, b) = 0$ hold for each $s \geqq 0$ if and only if $\lim_r^1 E_r^{s,s+i}(X^\bullet, b) = 0$ holds for each $s \geqq 0$. These combined conditions hold whenever: (i) $\{\pi_i(\mathrm{Tot}_s\, X^\bullet, b)\}$ can be topologized as a tower of compact Hausdorff groups (or linearly compact $\mathscr{C}$-groups [15]) and continuous homomorphisms; or (ii) for each $s \geqq 0$ there exists $r < \infty$ with $E_r^{s,s+i}(X^\bullet, b) = E_\infty^{s,s+i}(X^\bullet, b)$. Also for $i \geqq 1$ and $j \geqq 1$, the condition $E_{\infty+}^{s,s+i}(X^\bullet, b) = E_\infty^{s,s+i}(X^\bullet, b)$ holds for each $s \geqq j - 1$ whenever $\lim_r^1 E_r^{s,s+i}(X^\bullet, b) = 0$ holds for each $s \geqq j$. Next for $i = 0$, the condition $E_{\infty+}^{s,s}(X^\bullet, b) = E_\infty^{s,s}(X^\bullet, b)$ holds for each $s \geqq 0$ whenever $\{\pi_0\, \mathrm{Tot}_s\, X^\bullet\}$ can be topologized as a tower of compact Hausdorff spaces and continuous maps. Our remaining results for $i = 0$ may be slightly improved, using the modifications indicated in square brackets, when

$$[\pi_t X^\bullet, \pi_* X^\bullet] = 0 \qquad \text{for } 1 \leqq t \leqq r - 1$$

as in 2.6. Given $r \geqq 1$ [$r \geqq 2$], if $E_r^{s,s-1}(X^\bullet, b) = 0$ for all sufficiently large s, then $E_{\infty+}^{s,s}(X^\bullet, b) = E_\infty^{s,s}(X^\bullet, b)$ for all $s \geqq r$ [$s \geqq r - 1$]. Given $r \geqq 1$ and $k \geqq r + 1$ [$r \geqq 2$ and $k \geqq r$], if $E_r^{s,s}(X^\bullet, b)$ is finite for all $s \geqq k$, then

$$E_{\infty+}^{s,s}(X^\bullet, b) = E_\infty^{s,s}(X^\bullet, b) \qquad \text{for all } s \geqq k - 1.$$

When $X^\bullet$ is not termwise connected, it is convenient to focus on

4.4. Components of $X^\bullet$. By 10.7 there is a natural correspondence

$$(\pi_0\, \mathrm{Tot}_0\, X^\bullet)^{(1)} \approx \pi^0 \pi_0 X^\bullet = \{\alpha \in \pi_0 X^0 \mid d^0\alpha = d^1\alpha\}.$$

For each element $\alpha \in \pi^0\pi_0 X^\bullet$, let $X_\alpha^\bullet \subset X^\bullet$ be the cosimplicial subspace consisting of the connected components $X_\alpha^m \subset X^m$ at α, and let $X_c^\bullet = \amalg_\alpha X_\alpha^\bullet \subset X^\bullet$. Then $X_\alpha^\bullet$ and $X_c^\bullet$ are fibrant and

$$\coprod_{\alpha} \mathrm{Tot}_m X^\bullet_\alpha = \mathrm{Tot}_m X^\bullet_c = \mathrm{Tot}_m X^\bullet$$

for $1 \leqq m \leqq \infty$ where $\mathrm{Tot}_\infty = \mathrm{Tot}$. A vertex $b \in \mathrm{Tot}\, X^\bullet$ determines a connected component $X^\bullet_b$ with the same spectral sequence

$$\{E_r^{s,t}(X^\bullet_b, b)\} = \{E_r^{s,t}(X^\bullet, b)\} \qquad \text{for } (s,t) \neq (0,0),$$

with $\pi_i(\mathrm{Tot}\, X^\bullet_b, b) = \pi_i(\mathrm{Tot}\, X^\bullet, b)$ for $i > 0$, and with $\pi_0(\mathrm{Tot}\, X^\bullet_b, b) = F^1\pi_0(\mathrm{Tot}\, X^\bullet, b)$ where, more generally, $F^1\pi_0(\mathrm{Tot}_s X^\bullet, b)$ denotes the kernel of $\pi_0(\mathrm{Tot}_s X^\bullet, b) \to \pi_0(\mathrm{Tot}_0 X^\bullet, b)$. We say that $\{E_r^{s,t}(X^\bullet, b)\}$ *converges completely to* $F^1\pi_0(\mathrm{Tot}\, X^\bullet, b)$ *for* $t - s = 0$ when $E_{\infty+}^{s,s}(X^\bullet, b) = E_\infty^{s,s}(X^\bullet, b)$ for all $s \geqq 1$ and $F^1\pi_0(\mathrm{Tot}\, X^\bullet, b) \to \lim_s F^1\pi_0(\mathrm{Tot}_s X^\bullet, b)$ has trivial kernel. This is equivalent to saying that $\{E_r^{s,t}(X^\bullet_b, b)\}$ converges completely to $\pi_0(\mathrm{Tot}\, X^\bullet_b, b)$ for $t - s = 0$. Now, amplifying parts of 4.3, we obtain

4.5. E_2-criteria for complete convergence. For a given vertex $b \in \mathrm{Tot}\, X^\bullet$, suppose that each arbitrarily highly liftable vertex $a \in \mathrm{Tot}_1 X^\bullet$ with $a_0 = b_0 \in X^0$ is such that: (i) for each $i \geqq -1$, $\pi^s\pi_{s+i}(X^\bullet, a) = 0$ for all sufficiently large s; or (ii) $\pi^s\pi_{s+i}(X^\bullet, b)$ is finite for each $s, i \geqq 0$ except possibly for $(s,i) = (0,0)$. Then $\{E_r^{s,t}(X^\bullet, b)\}$ converges completely to $\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)$ for $t - s > 0$ and to $F^1\pi_0(\mathrm{Tot}\, X^\bullet, b)$ for $t - s = 0$; thus

$$\pi_i(\mathrm{Tot}\, X^\bullet, b) \approx \lim_s \pi_i(\mathrm{Tot}_s X^\bullet, b) \qquad \text{for each } i > 0,$$

$E_{\infty+}^{s,s+i}(X^\bullet, b) = E_\infty^{s,s+i}(X^\bullet, b)$ for each $s, i \geqq 0$ except possibly for $(s,i) = (0,0)$, and $F^1\pi_0(\mathrm{Tot}\, X^\bullet, b) \approx \lim_s F^1\pi_0(\mathrm{Tot}_s X^\bullet, b)$.

We conclude with a compact Hausdorff criterion for complete convergence, which will require some preliminaries. "Space" will temporarily mean "topological space". Let $f_0, f_1 : A \to B$ be continuous maps between compact Hausdorff spaces A and B, and let $\sim$ be the equivalence relation on B generated by the elementary equivalences $f_0(a) \sim f_1(a)$ for $a \in A$. Let $C = B/\sim$ and suppose that each equivalence class $c \in C$ has a representative $s_c \in B$ such that the members of c are all elementarily equivalent to s_c (i.e. for each $b \in c$ there exists $a \in A$ with $f_0(a) = b$ and $f_1(a) = s_c$). Then

LEMMA 4.6. *The quotient topology of C is compact Hausdorff, and the quotient map $B \to C$ is the coequalizer of $f_1, f_2 : A \to B$ in the category of compact Hausdorff spaces.*

PROOF. The compactness of C is immediate. By [17, p. 146] the quotient function $e : B \to C$ is an absolute coequalizer of $f_0, f_1 : A \to B$ in the category of sets, because there are obvious splitting functions $s: C \to B$ and $t: B \to A$ with $es = 1$, $f_0t = 1$, and $f_1t = se$. The lemma now follows from the proof in [17, pp. 153–155] that the forgetful functor from compact Hausdorff spaces to sets is monadic.

PROPOSITION 4.7. *Let L be a simplicial compact Hausdorff space which is fibrant as a simplicial set. Then $\pi_0 L$ is naturally a compact Hausdorff space and $\pi_n(L, v)$ is naturally a compact Hausdorff group for each $n > 0$ and vertex $v \in L$.*

PROOF. The quotient space $\pi_0 L$ of L_0 is compact Hausdorff by 4.6 since the elementary equivalences determined by $d_0, d_1 : L_1 \to L_0$ form an equivalence relation. Likewise, the quotient space $\pi_n(L, v)$ of $\tilde{L}_n = d_0^{-1}(v) \cap \cdots \cap d_n^{-1}(v)$ is compact Hausdorff by 4.6 applied to $d_0, d_1 : L'_{n+1} \to \tilde{L}_n$ where

$$L'_{n+1} = d_0^{-1}(\tilde{L}_n) \cap d_1^{-1}(\tilde{L}_n) \cap d_2^{-1}(v) \cap \cdots \cap d_{n+1}^{-1}(v).$$

Finally, the subtraction operation $\pi_n(L, v) \times \pi_n(L, v) \to \pi_n(L, v)$ is continuous since it is induced by the quotient maps $(d_1, d_0) : L''_{n+1} \to \pi_n(L, v) \times \pi_n(L, v)$ and $d_2 : L''_{n+1} \to \pi_n(L, v)$ where

$$L''_{n+1} = d_0^{-1}(\tilde{L}_n) \cap d_1^{-1}(\tilde{L}_n) \cap d_2^{-1}(\tilde{L}_n) \cap d_3^{-1}(v) \cap \cdots \cap d_{n+1}^{-1}(v).$$

We hope to further investigate compact homotopy theory, but we now combine 4.7 with 4.3 to give

4.8. A compact Hausdorff criterion for complete convergence. Suppose that $\{\mathrm{Tot}_s X^\bullet\}$ is weakly equivalent to a continuous tower of simplicial compact Hausdorff spaces which are fibrant as simplicial sets. Then, for each vertex $b \in \mathrm{Tot}\, X^\bullet$, $\{E_r^{s,t}(X^\bullet, b)\}$ converges completely to $\pi_{t-s}(\mathrm{Tot}\, X^\bullet, b)$ for each $t - s \geqq 0$. This applies, for instance, when $X^\bullet$ can be topologized as a cosimplicial simplicial compact Hausdorff space, or when $\pi_i(\mathrm{Tot}_s X^\bullet, v)$ is finite for each $i, s \geqq 0$ and each vertex $v \in \mathrm{Tot}_s X^\bullet$. In the latter case we may construct a weak equivalence from $\{\mathrm{Tot}_s X^\bullet\}$ to a tower of minimal fibrations of fibrant simplicial finite sets.

§5. Cosimplicial obstruction theory

Let $X^\bullet$ remain a fibrant cosimplicial space. We shall develop an obstruction theory for liftings of vertices and paths in the tower $\{\mathrm{Tot}_m X^\bullet\}_{m \geq 0}$.

5.1. The obstruction cocycle $c(b)$. By 10.7 for $m \geqq 0$, a vertex $b \in \mathrm{Tot}_m X^\bullet$ lifts to $\mathrm{Tot}_{m+1} X^\bullet$ if and only if the map $b^{m+1}: \dot{\Delta}^{m+1} \to X^{m+1}$ is nullhomotopic. Thus a vertex $b \in \mathrm{Tot}_0 X^\bullet = X^0$ lifts to $\mathrm{Tot}_1 X^\bullet$ if and only if $[b] \in \pi_0 X^0$ belongs to $\pi^0\pi_0 X^\bullet$. For a more detailed analysis when $m \geqq 1$, we define the *obstruction cocycle* $c(b) \in N\pi_m(X^{m+1}, b)$ to be the class of $b^{m+1}: (\dot{\Delta}^{m+1}, 0) \to (X^{m+1}, b)$ using the "reverse orientation" of $\dot{\Delta}^{m+1}$, i.e. using an equivalence $|(\dot{\Delta}^{m+1}, 0)| \simeq |(S^m, *)|$ making $[-\partial\iota_{m+1}] \in H_m\dot{\Delta}^{m+1}$ correspond to $[\iota_m] \in H_m(\Delta^m/\dot{\Delta}^m) = H_m S^m$. Thus

$$c(b) = [d^1b^1][d^0b^1]^{-1}[d^2b^1]^{-1} \qquad \text{when } m = 1.$$

In general by 10.7 and 12.12, $c(b) \in N\pi_m(X^{m+1}, v)$ has the properties:

(i) $c(b)$ is natural in $X^\bullet$ and in b, i.e. for cosimplicial maps and paths in $\mathrm{Tot}_m X^\bullet$.

(ii) $c(b) = 0$ if and only if b lifts to $\mathrm{Tot}_{m+1} X^\bullet$.

(iii) $c(b)$ lives to the highest term $E_t^{m+1,m}(X^\bullet, b)$ defined by 2.4, 2.5, or 2.6. Thus it lives to the term with $t = [(m+2)/2]$ in general, and with $t = [(m+3)/2]$ when $X^\bullet$ has vanishing Whitehead products.

We now introduce the higher order obstructions.

5.2. Obstructions to lifting vertices. For $m \geqq 0$ and $r \geqq 1$, suppose that a vertex $b \in \mathrm{Tot}_m X^\bullet$ is liftable to $\mathrm{Tot}_{m+r-1} X^\bullet$. Then the rth *order obstruction class* $\mathcal{O}_r(b) \subset N\pi_{m+r-1}(X^{m+r}, b)$ is the set of all $c(\tilde{b})$ for vertices $\tilde{b} \in \mathrm{Tot}_{m+r-1} X^\bullet$ lifting b. Clearly $b \in \mathrm{Tot}_m X^\bullet$ is liftable to $\mathrm{Tot}_{m+r} X^\bullet$ if and only if $0 \in \mathcal{O}_r(b)$. Now suppose $r \leqq m+1$ or $r = m+2$ with $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq 2m+1$ using the notation of 2.6. Then, by 5.1 and 5.3 below, $\mathcal{O}_r(b)$ is a coset forming an rth *order obstruction element* $\omega_r(b) \in E_r^{m+r,m+r-1}(X^\bullet, b)$ which has the properties:

(i) $\omega_r(b)$ is natural in $X^\bullet$ and b.

(ii) $\omega_r(b) = 0$ if and only if b is liftable to $\mathrm{Tot}_{m+r} X^\bullet$.

(iii) $\omega_r(b)$ lives to the highest term $E_t^{m+r,m+r-1}(X^\bullet, b)$ defined by 2.4, 2.5, or 2.6.

There is a convenient alternative version of this obstruction. For a vertex $a \in \mathrm{Tot}_n X^\bullet$ with $n \geqq 0$, suppose that $1 \leqq r \leqq (n+2)/2$ or $r = (n+3)/2$ with $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq n$. Then the rth *order obstruction element* $\gamma_t(a) \in E_r^{n+1,n}(X^\bullet, a)$ is defined by $\gamma_r(a) = \omega_r(a_{n-r+1})$. Clearly $\gamma_r(a) = 0$ if and only if the projection of $a \in \mathrm{Tot}_n X^\bullet$ to $\mathrm{Tot}_{n-r+1} X^\bullet$ lifts to $\mathrm{Tot}_{n+1} X^\bullet$. Moreover, when $\gamma_{r+1}(a) \in E_{r+1}^{n+1,n}(X^\bullet, b)$ is defined, it is given by $[\gamma_r(a)]$.

5.3. The difference cochain $D(a, J, b)$. For vertices $a, b \in \mathrm{Tot}_m X^\bullet$ with $m \geqq 1$ and a path $J: \Delta^1 \to \mathrm{Tot}_{m-1} X^\bullet$ from a_{m-1} to b_{m-1}, the *difference cochain* $D(a, J, b) \in N\pi_m(X^m, b)$ is represented by a map

$$(aJ)^m \amalg b^m : \Delta^m \amalg_{\dot{\Delta}^m} \Delta^m \to X^m$$

using the "left minus right" orientation, where $aJ \in \mathrm{Tot}_m X^\bullet$ is the endpoint of a path from $a \in \mathrm{Tot}_m X^\bullet$ lifting J. Equivalently, $D(a, J, b) = \Phi([a][J])$ using the bijection

$$\Phi : \pi_0 \mathrm{Fib}_m(X^\bullet, b) \approx N\pi_m(X^m, b)$$

of 10.2. Thus $D(a, J, b) = [d^1J]^{-1}[a^1][d^0J][b^1]^{-1}$ when $m = 1$. We denote $D(a, 1, b)$ by $D(a, b)$ when $a_{m-1} = b_{m-1}$. In general, by Sections 10–12, $D(a, J, b) \in N\pi_m(X^m, b)$ has the following properties:

(i) $D(a, J, b)$ is natural in $X^\bullet$ and in (a, J, b).

(ii) $D(a, J, b) = 0$ if and only if J lifts to a path from a to b.

(iii) If $a = b$ then $D(b, J, b) = \partial_*[J]$ using $\partial_* : \pi_1 \mathrm{Tot}_{m-1}(X^\bullet, b) \to N\pi_m(X^m, b)$ of 3.2.

(iv) $D(a, J, b)[K] + D(b, K, c) = D(a, JK, c)$ for each vertex $c \in \mathrm{Tot}_m X^\bullet$ and path $K: \Delta^1 \to \mathrm{Tot}_{m-1} X^\bullet$ from b_{m-1} to c_{m-1} where $[JK] = [J][K]$.

(v) If $m \geqq 2$ or $m = 1$ with $c(b)$ in the center of $\pi_1(X^2, b)$, then $\delta D(a, J, b) = c(b) - c(a)[J]$ in $N\pi_m(X^{m+1}, b)$. More generally, for $r \geqq 1$, suppose that $a, b \in \mathrm{Tot}_m X^\bullet$ lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}_{m+r-1} X^\bullet$, where $\bar{b}$ is sufficiently liftable so that $E_{r+1}^{m,m}(X^\bullet, \bar{b})$ is defined by 2.4, 2.5, or 2.6. Then $D(a, J, b)$ lives to $E_r^{m,m}(X^\bullet, \bar{b})$ and $d_r[D(a, J, b)] = [c(\bar{a})[J] - c(\bar{b})]$ in the target $E_t^{m+r,m+r-1}(X^\bullet, \bar{b})$ of the differential d_r on $E_r^{m,m}(X^\bullet, \bar{b})$. This target has $t = \min\{r, m\}$ in general, and has $t = \min\{r, m+1\}$ when $X^\bullet$ has vanishing Whitehead products.

(vi) For each element $\alpha \in N\pi_m(X^m, b)$ there exists a lifting $a' \in \mathrm{Tot}_m X^\bullet$ of $J(0) \in \mathrm{Tot}_{m-1} X^\bullet$ with $D(a', J, b) = \alpha$.

(vii) For $r \geqq 1$ suppose that $b \in \mathrm{Tot}_m X^\bullet$ lifts to a vertex $\bar{b} \in \mathrm{Tot}_{m+r-1} X^\bullet$ which is sufficiently liftable so that $E_{r+1}^{m,m}(X^\bullet, \bar{b})$ is defined by 2.4, 2.5, or 2.6. If $D(a, J, b)$ lives to $E_r^{m,m}(X^\bullet, \bar{b})$ and if $\alpha \in N\pi_{m+r-1}(X^{m+r}, \bar{b})$ is an element with $d_r[D(a, J, b)] = [\alpha]$ in the target $E_t^{m+r,m+r-1}(X^\bullet, \bar{b})$ of d_r, then $a \in \mathrm{Tot}_m X^\bullet$ lifts to a vertex $\bar{a} \in \mathrm{Tot}_{m+r-1} X^\bullet$ with $\alpha = c(\bar{a})[J] - c(\bar{b})$ in $N\pi_{m+r-1}(X^{m+r}, \bar{b})$.

5.4. Obstructions to lifting paths. For $m \geqq -1$ and $r \geqq 1$ with $m + r \geqq 1$, suppose that $a, b \in \mathrm{Tot}_{m+r} X^\bullet$ are vertices and $J: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ is a path from a_m to b_m liftable to a path from a_{m+r-1} to b_{m+r-1} in $\mathrm{Tot}_{m+r-1} X^\bullet$. Then the rth *order obstruction class* $\mathscr{D}_r(a, J, b) \subset N\pi_{m+r}(X^{m+r}, b)$ is the set of all $D(a, \bar{J}, b)$ for paths $\bar{J}: \Delta^1 \to \mathrm{Tot}_{m+r-1} X^\bullet$ from a_{m+r-1} to b_{m+r-1} lifting J. Clearly $J: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ is liftable to a path from a to b in $\mathrm{Tot}_{m+r} X^\bullet$ if and only if $0 \in \mathscr{D}_r(a, J, b)$. Now suppose that $a, b \in \mathrm{Tot}_{m+r} X^\bullet$ are liftable to $\mathrm{Tot}_{m+2r-1} X^\bullet$. Then, by 5.3 and 5.5 below, $\mathscr{D}_r(a, J, b)$ is a coset forming an rth *order obstruction element* $\nabla_r(a, J, b) \in E_r^{m+r, m+r}(X^\bullet, b)$ which has the properties:

(i) $\nabla_r(a, J, b)$ is natural in $X^\bullet$ and in (a, J, b).

(ii) $\nabla_r(a, J, b) = 0$ if and only if J is liftable to a path from a to b in $\mathrm{Tot}_{m+r} X^\bullet$.

(iii) If $a = b$ then $\nabla_r(b, J, b) = \partial_*[J]$ using $\partial_*: \pi_1 \mathrm{Tot}_m(X^\bullet, b)^{(r-1)} \to E_r^{m+r, m+r}(X^\bullet, b)$ of 3.3.

(iv) Let $c \in \mathrm{Tot}_{m+r} X^\bullet$ be another vertex liftable to $\mathrm{Tot}_{m+2r-1} X^\bullet$, and let $K: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ be a path from b_m to c_m liftable to a path from b_{m+r-1} to c_{m+r-1} in $\mathrm{Tot}_{m+r-1} X^\bullet$. If $m \geqq 0$ or if $m = -1$ with $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq r - 1$, then $\nabla_r(a, J, b)[K] + \nabla_r(b, K, c) = \nabla_r(a, JK, c)$.

(v) If $m \geqq 0$ or $m = -1$ with $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq r - 1$, then $d_r \nabla_r(a, J, b) = \omega_r(a)[J] - \omega_r(b)$ in $E_r^{m+2r, m+2r-1}(X^\bullet, b)$. If a and b lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}_{m+r+t-1} X^\bullet$ with $t \geqq r$, then $\nabla_r(a, J, b)$ lives to $E_t^{m+r, m+r}(X^\bullet, \bar{b})$; and if a and b lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}\, X^\bullet$, then $\nabla_r(a, J, b)$ lives to $E_{\infty+}^{m+r, m+r}(X^\bullet, \bar{b})$.

(vi) Let $K: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ be a path ending at b_m with lifting $\bar{K}: \Delta^1 \to \mathrm{Tot}_{m+r-1} X^\bullet$ ending at b_{m+r-1}. For an element $\alpha \in E_r^{m+r, m+r}(X^\bullet, b)$ there exists a vertex $c \in \mathrm{Tot}_{m+r} X^\bullet$ liftable to $\mathrm{Tot}_{m+2r-1} X^\bullet$ with $c_{m+r-1} = \bar{K}(0)$ and with $\nabla_r(c, K, b) = \alpha$.

As in 5.2, there is a convenient alternative version of this obstruction. For vertices $a, b \in \mathrm{Tot}\, X^\bullet$, let $K: \Delta^1 \to \mathrm{Tot}_n X^\bullet$ be a path from a_n to b_n with $n \geqq 0$, and suppose that $1 \leqq r \leqq n + 2$. Then the rth *order obstruction element* $\Delta_r(a, K, b) \in E_r^{n+1, n+1}(X^\bullet, b)$ is defined by $\Delta_r(a, K, b) = \nabla_r(a, K_{n-r+1}, b)$. Clearly $\Delta_r(a, K, b) = 0$ if and only if the projection of K to $\mathrm{Tot}_{n-r+1} X^\bullet$ lifts to a path from a_{n+1} to b_{n+1} in $\mathrm{Tot}_{n+1} X^\bullet$. The element $\Delta_r(a, K, b)$ always lifts to $E_{\infty+}^{n+1, n+1}(X^\bullet, b)$. Moreover, when $\Delta_{r+1}(a, K, b) \in E_{r+1}^{n+1, n+1}(X^\bullet, b)$ is defined, it is given by $[\Delta_r(a, K, b)]$. We have implicitly used

5.5. The difference cochain $D'(K, L)$. For $m \geqq 0$ let $K, L: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ be paths from a vertex $a \in \mathrm{Tot}_m X^\bullet$ to a vertex $b \in \mathrm{Tot}_m X^\bullet$ such that K and L project to the same path $J: \Delta^1 \to \mathrm{Tot}_{m-1} X^\bullet$. The *difference cochain* $D'(K, L) \in N\pi_{m+1}(X^m, b)$ is given by $\Phi([L^{-1}K])$ using the isomorphism

$$\Phi: \pi_1 \mathrm{Fib}_m(X^\bullet, b) \approx N\pi_{m+1}(X^m, b)$$

of 10.2, where the loop $L^{-1}K$ is homotoped to $\mathrm{Fib}_m(X^\bullet, b)$ over the canonical contraction of $J^{-1}J$. In general by Sections 10 and 11, $D'(K, L) \in N\pi_{m+1}(X^m, b)$ has the following properties:

(i) $D'(K, L)$ is natural in $X^\bullet$ and (K, L).

(ii) $D'(K, L) = 0$ if and only if K is path homotopic to L through liftings of J.

(iii) $D'(L, M) + D'(K, L) = D'(K, M)$ for each path $M: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ from a to b lifting J.

(iv) If $a, b \in \mathrm{Tot}_m X^\bullet$ respectively lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}_{m+1} X^\bullet$, then

$$\delta D'(K, L) = \partial_*[L^{-1}K] = -D(\bar{a}, L, \bar{b})[L^{-1}K] + D(\bar{a}, K, \bar{b})$$

in $N\pi_{m+1}(X^{m+1}\bar{b})$, where $D(\bar{a}, L, \bar{b})[L^{-1}K] = D(\bar{a}, L, \bar{b})$ when $m \geqq 1$. More generally for $r \geqq 1$ suppose that $a, b \in \mathrm{Tot}_m X^\bullet$ and $K, L: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ respectively lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}_{m+r} X^\bullet$ and paths $\bar{K}, \bar{L}: \Delta^1 \to \mathrm{Tot}_{m+r-1} X^\bullet$ from $\bar{a}_{m+r-1}$ to $\bar{b}_{m+r-1}$, where $\bar{b}$ is sufficiently liftable so that $E_{r+1}^{m,m+1}(X^\bullet, \bar{b})$ is defined by 2.4, 2.5, or 2.6. Then $D'(K, L)$ lives to $E_r^{m,m+1}(X^\bullet, \bar{b})$ and

$$d_r[D'(K, L)] = [\partial_*[\bar{L}^{-1}\bar{K}]] = [-D(\bar{a}, \bar{L}, \bar{b})[L^{-1}K] + D(\bar{a}, \bar{K}, \bar{b})]$$

in $E_r^{m+r,m+r}(X^\bullet, \bar{b})$, where $D(\bar{a}, L, \bar{b})[L^{-1}K] = D(\bar{a}, L, \bar{b})$ when $m \geqq 1$.

(v) For each element $\alpha \in N\pi_{m+1}(X^m, b)$, there exists a path $K': \Delta^1 \to \mathrm{Tot}_m X^\bullet$ from a to b lifting J with $D'(K', L) = \alpha$.

(vi) For $r \geqq 2$ suppose that $a, b \in \mathrm{Tot}_m X^\bullet$ and $L: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ respectively lift to vertices $\bar{a}, \bar{b} \in \mathrm{Tot}_{m+r} X^\bullet$ and path $\bar{L}: \Delta^1 \to \mathrm{Tot}_{m+r-1} X^\bullet$ from $\bar{a}_{m+r-1}$ to $\bar{b}_{m+r-1}$, where $\bar{b}$ is sufficiently liftable so that $E_{r+1}^{m,m+1}(X^\bullet, \bar{b})$ is defined by 2.4, 2.5, and 2.6. If $D'(K, L)$ lives to $E_r^{m,m+1}(X^\bullet, \bar{b})$ and if $\alpha \in N\pi_{m+r}(X^{m+r}, \bar{b})$ is an element with $d_r[D'(K, L)] = [\alpha]$ in $E_r^{m+r,m+r}(X^\bullet, \bar{b})$, then $K: \Delta^1 \to \mathrm{Tot}_m X^\bullet$ lifts to a path $\bar{K}: \Delta^1 \to \mathrm{Tot}_{m+r-1} X^\bullet$ from $\bar{a}_{m+r-1}$ to $\bar{b}_{m+r-1}$ with

$$\alpha = \partial_*[\bar{L}^{-1}\bar{K}] = -D(\bar{a}, \bar{L}, \bar{b})[L^{-1}K] + D(\bar{a}, \bar{K}, \bar{b})$$

in $N\pi_{m+r}(X^{m+r}, \bar{b})$, where $D(\bar{a}, \bar{L}, \bar{b})[L^{-1}K] = D(\bar{a}, \bar{L}, \bar{b})$ when $m \geqq 1$.

§6. Connectivity and comparison results

For a fibrant cosimplicial space $X^\bullet$, we now apply our homotopy spectral sequence and obstruction machinery to derive some connectivity and comparison results for Tot $X^\bullet$.

6.1. Nonemptyness of Tot $X^\bullet$. If $X^\bullet$ is pointed, or has a nonempty augmentation, then clearly Tot $X^\bullet$ is nonempty. In general by 4.4, Tot $X^\bullet$ is the disjoint union of the Tot $X^\bullet_\alpha$ for $\alpha \in \pi^0\pi_0 X^\bullet \approx (\pi_0 \operatorname{Tot}_0 X^\bullet)^{(1)}$, so the connected components $X^\bullet_\alpha \subset X^\bullet$ may be inspected individually. Moreover, by 14.4, there is a bijection $\pi^1\pi_1^{gd} X^\bullet_\alpha \approx (\pi_0 \operatorname{Tot}_1 X^\bullet_\alpha)^{(1)}$ so $\pi_0 \operatorname{Tot}_2 X^\bullet_\alpha$ is nonempty iff $\pi^1\pi_1^{gd} X^\bullet_\alpha$ is nonempty. Various other nonemptyness results follow by obstruction theory. For instance, for $r \geqq 1$ if a vertex $b \in \operatorname{Tot}_{r-1} X^\bullet$ lifts to $\operatorname{Tot}_{2r-2} X^\bullet$ and if $E_r^{k,k-1}(X^\bullet, b) = 0$ for all $k \geqq 2r-1$, then b lifts to Tot $X^\bullet$. Likewise, for $r \geqq 2$ if a vertex $b \in \operatorname{Tot}_{r-2} X^\bullet$ lifts to $\operatorname{Tot}_{2r-3} X^\bullet$ where $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq 2r-3$ and if $E_r^{k,k-1}(X^\bullet, b) = 0$ for all $k \geqq 2r-2$, then b lifts to Tot $X^\bullet$. On the other hand, if $X^\bullet \simeq Sk_m \mathbf{\Delta}$ then $\operatorname{Tot}_k X^\bullet$ is empty for $k > m$.

6.2. Connectivity of Tot $X^\bullet$. When Tot $X^\bullet$ is nonempty we may form $\{E_r(X^\bullet, b)\}$ at a vertex $b \in \operatorname{Tot} X^\bullet$ and use convergence results of Section 4 or [8] to study $\pi_*(\operatorname{Tot} X^\bullet, b)$. Thus, for $b \in \operatorname{Tot} X^\bullet$, $m \geqq 0$, and $r \geqq 1$, if $E_r^{s,s+i}(X^\bullet, b) = 0$ whenever $s \geqq 0$ and $0 \leqq i \leqq m$, then Tot $X^\bullet$ is m-connected.

6.3. A comparison theorem. Let $f: X^\bullet \to Y^\bullet$ be a map of fibrant cosimplicial spaces. It is well-known [8, p. 277] that if $f: X^s \simeq Y^s$ for each $s \geqq 0$, then $\operatorname{Tot} f: \operatorname{Tot} X^\bullet \simeq \operatorname{Tot} Y^\bullet$. By 3.3 and 5.2, this conclusion follows using much weaker hypotheses at the E_r-level. Suppose $r \geqq 1$ and

$$f_*: (\pi_0 \operatorname{Tot}_{r-1} X^\bullet)^{(r-1)} \approx (\pi_0 \operatorname{Tot}_{r-1} X^\bullet)^{(r-1)}.$$

For each $[b] \in (\pi_0 \operatorname{Tot}_{r-1} X^\bullet)^{(r-1)}$ suppose that $f_*: E_r^{s,s+i}(X^\bullet, b) \to E_r^{s,s+i}(Y^\bullet, fb)$ is: (i) mono for $i \geqq -1$ and $s \geqq 2r-1$; (ii) iso for $i = 0$ and $s \geqq r$; and (iii) iso for $i \geqq 1$ and $s \geqq 0$. Then

$$f_*: (\pi_0 \operatorname{Tot}_m X^\bullet)^{(r-1)} \approx (\pi_0 \operatorname{Tot}_m Y^\bullet)^{(r-1)} \qquad \text{for each } m \geqq r-1,$$

and

$$f_*: \pi_i(\operatorname{Tot}_m X^\bullet, b)^{(r-1)} \approx \pi_i(\operatorname{Tot}_m Y^\bullet, fb)^{(r-1)}$$

$$\text{for each vertex } b \in \operatorname{Tot}_{m+2r-2} X^\bullet \qquad \text{with } m \geqq 0 \quad \text{and} \quad i \geqq 1.$$

Consequently, $\{\text{Tot}_s f\}: \{\text{Tot}_s X^\bullet\} \to \{\text{Tot}_s Y^\bullet\}$ is a weak prohomotopy equivalence (see [6, 8.5]) and $\text{Tot} f: \text{Tot}\, X^\bullet \simeq \text{Tot}\, Y^\bullet$.

Alternatively, suppose $r \geqq 2$ and $f_*: (\pi_0 \text{Tot}_{r-2} X^\bullet)^{(r-1)} \approx (\pi_0 \text{Tot}_{r-2} X^\bullet)^{(r-1)}$ with $[\pi_t X^\bullet, \pi_* X^\bullet] = 0$ for $1 \leqq t \leqq 2r-3$. For each $[b] \in (\pi_0 \text{Tot}_{r-2} X^\bullet)^{(r-1)}$ suppose that $f_*: E_r^{s,s+i}(X^\bullet, b) \to E_r^{s,s+i}(Y^\bullet, fb)$ is: (i) mono for $i \geqq -1$ and $s \geqq 2r-2$; (ii) iso for $i = 0$ and $s \geqq r-1$; and (iii) iso for $i \geqq 1$ and $s \geqq 0$. Then

$$f_*: (\pi_0 \text{Tot}_m X^\bullet)^{(r-1)} \approx (\pi_0 \text{Tot}_m Y^\bullet)^{(r-1)} \qquad \text{for each } m \geqq r-2,$$

and

$$f_*: \pi_i(\text{Tot}_m X^\bullet, b)^{(r-1)} \approx \pi_i(\text{Tot}_m Y^\bullet, fb)^{(r-1)}$$

$$\text{for each vertex } b \in \text{Tot}_{m+2r-2} X^\bullet \qquad \text{with } m \geqq 0 \quad \text{and} \quad i \geqq 1.$$

Consequently, $\text{Tot} f: \text{Tot}\, X^\bullet \simeq \text{Tot}\, Y^\bullet$ as above. For instance, *iff*: $X^\bullet \to Y^\bullet$ *is a map of fibrant, termwise simple, cosimplicial spaces such that* $f_*: \pi^s\pi_t X^\bullet \simeq \pi^s\pi_t Y^\bullet$ *is mono for* $t = s-1 \geqq 1$ *and iso for* $t \geqq s \geqq 0$, *then* $\text{Tot} f: \text{Tot}\, X^\bullet \simeq \text{Tot}\, Y^\bullet$. This instance also follows from

6.4. A simple derived homotopy exact sequence. Let $X^\bullet$ be a termwise simple fibrant cosimplicial space. Then for $m \geqq 1$ there is an exact sequence

$$(\pi_0 \text{Tot}_m X^\bullet)^{(1)} \xrightarrow{j} (\pi_0 \text{Tot}_{m-1} X^\bullet)^{(1)} \xrightarrow{\omega_2} \pi^{m+1}\pi_m X^\bullet$$

where j is the tower map and ω_2 is the lifting obstruction of 5.2, and there is a natural left action by the group $\pi^m\pi_m X^\bullet$ on the set $(\pi_0 \text{Tot}_m X^\bullet)^{(1)}$ such that elements of $(\pi_0 \text{Tot}_m X^\bullet)^{(1)}$ are in the same orbit iff they have the same image in $(\pi_0 \text{Tot}_{m-1} X^\bullet)^{(1)}$. For each vertex $b \in \text{Tot}_{m+1} X^\bullet$ and element $u \in \pi^m\pi_m X^\bullet \approx \pi^m\pi_m(X^\bullet, b)$, this action produces an element

$$u + [b] \in (\pi_0 \text{Tot}_m X^\bullet)^{(1)} \approx \pi_0(\text{Tot}_m X^\bullet, b)^{(1)}$$

which equals the image of u within the derived homotopy exact sequence

$$\cdots \to \pi_1(\text{Tot}_{m-1} X^\bullet, b)^{(1)} \to \pi_1(\text{Tot}_{m-2} X^\bullet, b)^{(1)}$$
$$\to \pi^m\pi_m(X^\bullet, b) \to \pi_0(\text{Tot}_m X^\bullet, b)^{(1)} \to \pi_0(\text{Tot}_{m-1} X^\bullet, b)$$

of 3.3. Furthermore, there exists a vertex $c \in \text{Tot}_{m+1} X^\bullet$ with $u + [b] = [c]$, $c_{m-1} = b_{m-1}$, and $\nabla_2(c, 1, b) = u$ by 5.4. The ordinary obstruction theoretic Hopf–Whitney–Eilenberg classification theorem now generalizes to

6.5. A simple classification theorem. Let $X^\bullet$ remain a termwise simple fibrant cosimplicial space, and for some $q \geqq 1$ suppose that: $\pi^s\pi_{s-1}X^\bullet = 0$ for all $s \geqq 2$; $\pi^s\pi_s X^\bullet = 0$ for all $s \neq q$; and $\pi^s\pi_{s+1}X^\bullet = 0$ whenever $0 \leqq s \leqq q-2$. Then there are natural bijections

$$\pi_0 \operatorname{Tot} X^\bullet \approx (\pi_0 \operatorname{Tot}_q X^\bullet)^{(1)} \approx \pi^q\pi_q X^\bullet.$$

§7. A classical homotopy spectral sequence

Before turning to the unstable Adams spectral sequence, we discuss a classical homotopy spectral sequence and associated obstructions for unpointed mapping spaces [4], [13]. Our discussion can easily be adapted to the pointed case.

7.1. The cosimplicial setup. For a space K and fibrant space L, we form the fibrant cosimplicial space $\operatorname{Map}^\bullet(K, L)$ where $\operatorname{Map}^m(K, L)$ is a product of copies of L indexed by the m-simplices of K. As in [8, p. 271],

$$\operatorname{Tot} \operatorname{Map}^\bullet(K, L) = \operatorname{Map}(K, L) \quad \text{and} \quad \{\operatorname{Tot}_s \operatorname{Map}^\bullet(K, L)\} = \{\operatorname{Map}(Sk_s K, L)\}.$$

The connected components (4.4) of $\operatorname{Map}^\bullet(K, L)$ correspond to members of

$$[Sk_0 K, L]^{(1)} \approx \pi^0\pi_0 \operatorname{Map}^\bullet(K, L) \approx [\pi_0 K, \pi_0 L]$$

where $[\pi_0 K, \pi_0 L]$ consists of functions $\pi_0 K \to \pi_0 L$, and as in 14.4

$$[Sk_1 K, L]^{(1)} \approx \pi^1\pi_1^{gd} \operatorname{Map}^\bullet(K, L) \approx [\pi_1^{gd} K, \pi_1^{gd} L]$$

where $[\pi_1^{gd} K, \pi_1^{gd} L]$ consists of the functors $\pi_1^{gd} K \to \pi_1^{gd} L$ modulo natural equivalences. For a map $b: Sk_1 K \to L$ extendable to $Sk_2 K$, $\pi^s\pi_t(\operatorname{Map}^\bullet(K, L), b)$ is given by $[\pi_0 K, \pi_0 L]$ for $(s, t) = (0, 0)$ and by the twisted cohomology $H^s(K; \pi_t L)_{b_*}$ associated with $b_*: \pi_1^{gd} K \to \pi_1^{gd} L$ for $s \geqq 0$ and $t \geqq 1$.

7.2. The classical homotopy spectral sequence and obstructions. For a map $b: K \to L$, we obtain the classical homotopy spectral sequence $\{E_r^{s,t}(\operatorname{Map}^\bullet(K, L), b)\}$ from 2.4 with

$$E_2^{s,t}(\operatorname{Map}^\bullet(K, L), b) = H^s(K; \pi_t L)_{b_*} \qquad \text{for } s \geqq 0 \quad \text{and} \quad t \geqq 1;$$

more generally, for a map $b: Sk_m K \to L$ extendable over $Sk_{2m} K$ with $m \geqq 1$, we obtain a truncated version of this spectral sequence defined for $1 \leqq r \leqq m+1$. Each groupoid map $\pi_1^{gd} K \to \pi_1^{gd} L$ is induced by a suitable map $Sk_2 K \to L$. By 5.2, for any map $b: Sk_n K \to L$ with $n \geqq 2$, there are obstructions $\gamma_r(b) \in E_r^{n+1,n}(\operatorname{Map}^\bullet(K, L), b)$ for $1 \leqq r \leqq (n+2)/2$ which vanish iff $b \mid Sk_{n-r+1} K$ extends over $Sk_{n+1} K$. These generalize the classical obstruction $\gamma_2(b) \in$

$H^{n+1}(K; \pi_n L)_{b*}$. Finally by 5.4, for maps $a, b: K \to L$ and a homotopy $H: \Delta^1 \times Sk_s K \to L$ from $a \mid Sk_s K$ to $b \mid Sk_s K$ with $s \geqq 0$, there are obstructions $\Delta_r(a, H, b) \in E_r^{s+1,s+1}(\mathrm{Map}^\bullet(K, L), b)$ for $1 \leqq r \leqq s+2$ which vanish iff $H \mid (\Delta^1 \times Sk_{s-r+1} K)$ lifts to a homotopy $H: \Delta^1 \times Sk_{s+1} K \to L$ from $a \mid Sk_{s+1} K$ to $b \mid Sk_{s+1} K$. These generalize the classical obstruction $\Delta_2(a, H, b) \in H^{s+1}(K; \pi_{s+1} L)_{b*}$.

7.3. Convergence. Let $b: K \to L$ be a fixed map. For each groupoid map $\pi_1^{gd} K \to \pi_1^{gd} K$ agreeing with b on vertices and for each $i \geqq -1$, suppose that the associated twisted cohomology $H^s(K; \pi_{s+i} L)$ vanishes for all sufficiently large s. Then, by 4.5, $\{E_r^{s,t}(\mathrm{Map}^\bullet(K, L), b)\}$ converges completely to $\pi_{t-s}(\mathrm{Map}(K, L), b)$ for $t - s > 0$ and to the kernel $F^1 \pi_0(\mathrm{Map}(K, L), b)$ of $[K, L] \to [\pi_0 K, \pi_0 L]$ over $b_*: \pi_0 K \to \pi_0 L$ for $t - s = 0$. This applies, for instance, when K is finite dimensional or L is a Postnikov space.

Next let $b: K \to L$ be a map where L is equivalent to a simplicial compact Hausdorff space whose underlying simplicial set is fibrant. Then, by 4.8, $\{E_r^{s,t}(\mathrm{Map}^\bullet(K, L), b)\}$ converges completely to $\pi_{t-s}(\mathrm{Map}(K, L), b)$ for $t - s \geqq 0$. This applies, for instance, when L is an F_p-completion, $L \simeq F_{p\infty} Y$, for $H_*(Y; F_p)$ of finite type [8].

Finally suppose that L is Q-nilpotent with homotopy groups of finite rank. Then by [15], for each $b: K \to L$ and $i \geqq 1$, $\{\pi_i(\mathrm{Map}(Sk_s K, L), b)\}$ is a continuous tower of linearly compact HQ-local groups. Thus by 4.3, $\{E_r^{s,t}(\mathrm{Map}^\bullet(K, L), b)\}$ converges completely to $\pi_{t-s}(\mathrm{Map}(K, L), b)$ for $t - s > 0$ and $[K, L] \approx \lim_s [Sk_s K, L]$.

7.4. The associated homology spectral sequences. By 2.7 there is a Hurewicz map from the homotopy spectral sequence of $\mathrm{Map}^\bullet(X, Y)$ to the corresponding homology spectral sequence over a commutative ring R. The latter is Anderson's spectral sequence ([1], [6, 4.2]) for $H_*(\mathrm{Map}(X, Y); R)$.

§8. Derived functors of derivations over the Steenrod algebra

Before discussing the unstable Adamas spectral sequence for mapping spaces in Section 9, we develop some algebraic preliminaries.

8.1. Discrete coalgebras over a field k. For a set W, let kW be the coalgebra consisting of the free k-module on W with comultiplication $\Delta: kW \to kW \otimes kW$ and counit $\varepsilon: kW \to k$ determined by $\Delta(w) = w \otimes w$ and $\varepsilon(w) = 1$ for $w \in W$. For a coalgebra C over k, let $\pi_0 C$ be the set of all $c \in C$ with $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. Then $k(\ \ )$ is left adjoint to $\pi_0(\ \ )$, with adjunction bijections $W \approx \pi_0 kW$ and injections $k\pi_0 C \subset C$ by, e.g., [22, p. 57]. A coalgebra C is called *discrete* when $k\pi_0 C = C$. The category of discrete coalgebras is clearly equivalent to the category of sets. Each coalgebra C is the direct limit of its finite dimensional subcoalgebras by, e.g. [22, p. 47]. Thus a commutative coalgebra C over the prime field $k = F_p$ has a natural Frobenius endomorphism $\xi : C \to C$ dual to the pth power endomorphism. *A commutative coalgebra C over F_p is discrete if and only if* $\xi = 1 : C \to C$, since a finite dimensional commutative algebra A over F_p has $(\ \ )^p = 1 : A \to A$ if and only if there is an algebra isomorphism $A \approx F_p \times \cdots \times F_p$.

8.2. The category CA. We let CA denote the category of unstable graded commutative coalgebras over the mod-p Steenrod algebra for a fixed prime p, exactly as in [7, 11.3] except that our present objects $B \in \mathrm{CA}$ need not be connected but must have $B_n = 0$ for $n < 0$ and B_0 discrete. Thus $H_*(Y; F_p) \in \mathrm{CA}$ for any space Y. Each object $B \in \mathrm{CA}$ decomposes canonically as a direct sum $B = \oplus_b B_b$ of connected subobjects $B_b \subset B$ for $b \in \pi_0 B_0$, where B_b is the image of the idempotent $e_b : B \to B$ given by the composition

$$B \xrightarrow{\Delta} B \otimes B \xrightarrow{u_b \otimes 1} F_p \otimes B = B$$

with u_b projecting to the summand $F_p = F_p b$ of $B_0 = F_p \pi_0 B_0$. Moreover, the maps in CA clearly carry components to components. The image of a map in CA is also in CA, and the category CA has arbitrary small colimits and limits. More specifically, a colimit (e.g. coproduct, coequalizer, etc.) in CA is the colimit of the underlying graded vector spaces with induced CA-structure; a finite product is the tensor product, while an infinite product may be constructed using cofree resolutions (8.3); and an equalizer of maps $\theta, \varphi : B \to C$ in CA is the largest subobject $E \subset B$ with $\theta | E = \varphi | E$, which is given by the image of $\oplus_\alpha E_\alpha \to B$ using all subobjects $E_\alpha \subset B$ with $\theta | E_\alpha = \varphi | E_\alpha$. In general, this equalizer $E \subset B$ map be smaller than the graded vector space $\ker(\theta - \phi) \subset B$; but when $\theta(b_1) = \varphi(b_2)$ implies $b_1 = b_2$ for $b_1, b_2 \in B$, then $E = \ker(\theta - \phi)$, because $\ker(\theta - \phi) \otimes \ker(\theta - \phi)$ will equal $\ker(\theta \otimes \theta - \varphi \otimes \varphi)$. Thus for a cosimplicial object $Y^\bullet$ over CA, the equalizer of $d^0, d^1 : Y^0 \to Y^1$ in CA is $\pi^0 Y = \ker(d^0 - d^1)$.

8.3. Derived functors on CA. Let Vect be the category of graded F_p-vector spaces W with $W_n = 0$ for $n < 0$. Extending [7, 11.4], the forgetful functor $J : \mathrm{CA} \to \mathrm{Vect}$ has a right adjoint $G : \mathrm{Vect} \to \mathrm{CA}$ with

$$G(W)=H_*\left(\prod_{n=0}^{\infty} K(W_n, n); F_p\right).$$

An object $B \in \mathrm{CA}$ is called *cofree* if each connected component $B_b \subset B$ is isomorphic to some value of G. Equivalently, an object $B \in \mathrm{CA}$ is cofree if and only if B is a retract of some $G(W)$. A *cosimplicial cofree resolution* of $B \in \mathrm{CA}$ consists of an augmented cosimplicial object $B \to Y^\bullet$ with Y^s cofree for $s \geqq 0$, with $B \to \pi^0 Y^\bullet$ iso, and with $\pi^s Y^\bullet = 0$ for $s \geqq 1$. One such resolution $B \to G^\bullet B$ with $(G^\bullet B)^s = G^{s+1}B$ is obtained by iterating the adjunction triple $G = GJ: \mathrm{CA} \to \mathrm{CA}$. Now as in [5, App.], each function $T: \mathrm{CA} \to M$ to an abelian category M has right derived functors $R^sT: \mathrm{CA} \to M$ for $s \geqq 0$ given by

$$(R^sT)(B) = \pi^s T(G^\bullet B) \approx \pi^s T(Y^\bullet) \qquad \text{for } B \in \mathrm{CA}$$

and for any cosimplicial cofree resolution $B \to Y^\bullet$. More generally, on the category $\mathrm{CA} \backslash B$ of objects under some $B \in \mathrm{CA}$, a functor $T: \mathrm{CA}\backslash B \to M$ has right derived functors $R^sT: \mathrm{CA}\backslash B \to M$ for $s \geqq 0$ defined as above by viewing $B \to G^\bullet B$ and $B \to Y^\bullet$ as cosimplicial objects over $\mathrm{CA}\backslash B$. This is justified by [5, App.] using the adjoint functors $J: \mathrm{CA}\backslash B \leftrightarrow \mathrm{Vect}\backslash JB : G$.

REMARK 8.4. For a more detailed study of such derived functors, one may use Quillen's machinery [20], [21]. By the dual of Theorem 4(*) in [20, II §4], there is a closed model category structure on the category $\nabla\mathrm{CA}$ of cosimplicial objects over CA, where a map $\phi: Y^\bullet \to Z^\bullet$ is: a *weak equivalence* when $\phi: \pi^* Y^\bullet \approx \pi^* Z^\bullet$; a *cofibration* when $Nf: NY^s \to NZ^s$ is monic for $s > 0$; and a *fibration* when f has the right lifting property for all weak equivalence-cofibrations. Now the above derived functors R^sT are constructed using a weak equivalence-cofibration $B \to Y^\bullet$ with $Y^\bullet$ fibrant.

8.5. Derivations in CA. As in [19, Corr.], let V denote the category of right modules M over the mod-p Steenrod algebra with $M_n = 0$ for $n \leqq 0$ and with the modified unstable condition $xP^t = 0$ for $|x| \leqq 2pt$ when p odd and $xSq^t = 0$ for $|x| \leqq 2t$ when $p = 2$. For $B \in \mathrm{CA}$, let $\mathrm{V}B$ denote the category of B-comodules $M \in \mathrm{V}$ with $\Delta_M: M \to B \otimes M$ respecting the right Steenrod action. Note that $\mathrm{V}B$ is an abelian category with enough injectives of the form $B \otimes \tilde{F}W$ for $W \in \mathrm{Vect}$ where $\tilde{F}: \mathrm{Vect} \to \mathrm{V}$ is right adjoint to the forgetful functor. A *derivation* from $M \in \mathrm{V}B$ to $B \in \mathrm{CA}$ is a Steenrod module homomorphism $D: M \to B$ such that $\Delta D: M \to B \otimes B$ equals $(1 \otimes D + \tau(D \otimes 1))\Delta_M$. Let $\mathrm{Der}_{\mathrm{CA}}(M, B)$ denote the F_p-module of such derivations. Each $M \in \mathrm{V}B$ determines an object $\imath(M) = B \oplus M$ in CA under B with comultiplication acting on M by $\Delta_M + \tau\Delta_M : M \to (B \otimes M) \oplus (M \otimes B)$, and this produces a functor $\iota : VB \to CA \backslash B$. For a map $\phi : B \to C$ in CA and $M \in VB$, there is a natural isomorphism

$$\mathrm{Hom}_{CA\backslash B}(\iota M, C) \approx \mathrm{Der}_{CA}(M, C)$$

where M is given the C-comodule structure induced by ϕ.

8.6. Another approach to derivations in CA. For $B \in CA$, VB is in fact equivalent to the category of abelian cogroup objects in $CA \backslash B$, and there is an abelianization functor $Ab_B : CA \backslash B \to VB$ right adjoint to $\iota : VB \to CA \backslash B$ as explained below. Thus there is a natural isomorphism

$$\mathrm{Hom}_{VB}(M, Ab_B C) \approx \mathrm{Der}_{CA}(M, C)$$

for $f : B \to C$ in CA and $M \in VB$. The functor $Ab_B : CA \backslash B \to VB$ carries $f : B \to C$ to the object $Ab_B C$ in VB given by the kernel of the map

$$B \otimes \Delta - (B \otimes f \otimes C)(\Delta \otimes C) - (B \otimes \tau)(B \otimes f \otimes C)(\Delta \otimes C)$$

from $B \otimes C$ to $B \otimes C \otimes C$ in the category UB of unstable B-comodules over the Steenrod algebra. This follows since the composition

$$Ab_B(C) \subset B \otimes C \xrightarrow{\varepsilon \otimes C} F_p \otimes C = C$$

is the universal example of a derivation to $C \in CA$ from a VB object with VC-structure induced by $\phi : B \to C$. As a B-comodule, $Ab_B C$ is independent of Steenrod actions. Moreover,

$$Ab_B C \approx B \square_C Ab_C C$$

where $\square_C$ is the cotensor product, and $Ab_C C$ is a coalgebraic analogue of the "Kahler module of differentials." Finally, $Ab_B C$ is easily determined in special cases: (i) if $B = F_p$, then $Ab_B C \in V$ consists of the primitives in the component of C at B; and (ii) if $C = G(W)$ for some $W \in \mathrm{Vect}$, then $Ab_B C \approx B \otimes \tilde{F} W$ in VB.

8.7. Derived functors of derivations in CA. For a map $\phi : B \to C$ in CA, $s \geqq 0$, and $t \geqq 1$, we form the F_p-module

$$\mathrm{Der}^{s,t}_{CA}(B, C)_\phi = \pi^s \,\mathrm{Der}_{CA}(\tilde{H}_* S^t \otimes B, G^\bullet C)_\phi$$

$$= \pi^s \,\mathrm{Hom}_{CA\backslash B}(H_* S^t \otimes B, G^\bullet C)_\phi$$

where $\tilde{H}_*$ and H_* denote F_p-homology and where the optional subscript ϕ

indicates the dependence of structures on $\phi : B \to C$. By 8.3, we can use an arbitrary cosimplicial cofree resolution $C \to Y^\bullet$ in place of $C \to G^\bullet C$. Since $C \to Y^0$ is the equalizer of $d^0, d^1 : Y^0 \to Y^1$ in CA, we have

$$\begin{aligned}\mathrm{Der}^{0,t}_{\mathrm{CA}}(B, C)_\phi &= \mathrm{Der}_{\mathrm{CA}}(\tilde{H}_* S^t \otimes B, C)_\phi \\ &= \mathrm{Hom}_{\mathrm{CA}\backslash B}(H_* S^t \otimes B, C)_\phi\end{aligned}$$

for $t \geqq 1$.

8.8. A spectral sequence for $\mathrm{Der}^{s,t}_{\mathrm{CA}}(B, C)_\phi$. For any factorization of $\phi : B \to C$ by maps $B \to K$ and $K \to C$ in CA, there is a convergent cohomological spectral sequence

$$E_2^{s,q} = \mathrm{Ext}^s_{\mathrm{V}K}(\tilde{H}_* S^t \otimes B, R^q Ab_K C) \Rightarrow \mathrm{Der}^{s+q,t}_{\mathrm{CA}}(B, C)_\phi$$

for $t \geqq 1$, constructed using the isomorphism

$$\mathrm{Der}_{\mathrm{CA}}(\tilde{H}_* S^t \otimes B, G^\bullet C)_\phi \approx \mathrm{Hom}_{\mathrm{V}K}(H_* S^t \otimes B, Ab_K G^\bullet C).$$

The K-comodule $R^q Ab_K C$ is independent of Steenrod actions, and the results of [3] and [21] can be applied. When $\phi : B \to C$ is trivial and $K = F_p$, $R^q Ab_K C$ becomes a derived functor of primitives, and we recover the spectral sequence of Miller [19, 2.5].

8.9. On $\mathrm{Der}^{s,t}_{\mathrm{CA}}(B, C)_\phi$ **for a Hopf algebra** C. Suppose that C is a group object in CA; that is, $C \in \mathrm{CA}$ is equipped with a multiplication map $C \otimes C \to C$, a unit map $F_p \to C$, and an antipode map $C \to C$ in CA satisfying the group conditions. For example, C might be $H_* G$ for a topological group G or $U(M)$ for an unstable right-module M over the Steenrod algebra. Then for any object B and maps $\theta, \phi : B \to C$ in CA, there is a canonical isomorphism

$$\mathrm{Der}^{s,t}_{\mathrm{CA}}(B, C)_\theta \approx \mathrm{Der}^{s,t}_{\mathrm{CA}}(B, C)_\phi$$

for $s \geqq 0$ and $t \geqq 1$; and thus a given $\phi : B \to C$ may be replaced by a trivial map. This follows using the cosimplicial pairing $G^\bullet C \otimes G^\bullet C \to G^\bullet C$ induced by the multiplication map $C \otimes C \to C$ and by the natural pairing

$$GD \otimes GE \to G(D \otimes E) \qquad \text{for } D, E \in \mathrm{CA}.$$

Finally, in the Massey–Peterson case of a map $\phi : B \to UM$ in CA for an unstable right-module M over the Steenrod algebra, we have

$$\mathrm{Der}^{s,t}_{\mathrm{CA}}(B, UM)_\phi \approx \mathrm{Ext}^s_{\mathrm{U}}(\tilde{H}_* S^t \otimes B, M)$$

as in [7, 13.4] for $s \geqq 0$ and $t \geqq 1$, where U is the category of unstable right-modules over the Steenrod algebra.

We refer the reader to [16, §6] for another approach to $\mathrm{Der}_{\mathrm{CA}}^{s,t}(B, C)$ in an important special case. Finally we briefly discuss

8.10. Homological Lannes functors and their derived functors. For $B \in \mathrm{CA}$ the functor $(-)\otimes B : \mathrm{CA} \to \mathrm{CA}$ has a right adjoint $\mathrm{Map}_{\mathrm{CA}}(B, -) : \mathrm{CA} \to \mathrm{CA}$ constructed as follows: for $W \in \mathrm{Vect}$,

$$\mathrm{Map}_{\mathrm{CA}}(B, GW) = G\,\mathrm{Map}_{\mathrm{Vect}}(B, W)$$

where $\mathrm{Map}_{\mathrm{Vect}}(B, W)$ is $\{\mathrm{Hom}_{\mathrm{Vect}}(\Sigma^i B, W)\}_{i \geq 0}$; and for any $C \in \mathrm{CA}$, $\mathrm{Map}_{\mathrm{CA}}(B, C)$ is the equalizer in CA of

$$d^0, d^1 : \mathrm{Map}_{\mathrm{CA}}(B, GC) \to \mathrm{Map}_{\mathrm{CA}}(B, GGC).$$

The resulting *Lannes functor* $\mathrm{Map}_{\mathrm{CA}}(B, C)$, whose cohomological version is in [16], turns CA into a cartesian closed category [17, p. 95]. There are right derived functors

$$\mathrm{Map}_{\mathrm{CA}}^{s,t}(B, C) = \pi^s \mathrm{Map}_{\mathrm{CA}}(B, G^\bullet C)_t$$

for $s \geqq 0$ with $\mathrm{Map}_{\mathrm{CA}}^{0,*}(B, C) = \mathrm{Map}_{\mathrm{CA}}(B, C)$. Moreover, for a map $\phi : B \to C$ in CA, $s \geqq 0$, and $t \geqq 1$, there is a natural homomorphism

$$h : \mathrm{Der}_{\mathrm{CA}}^{s,t}(B, C)_\phi \to \mathrm{Map}_{\mathrm{CA}}^{s,t}(B, C)$$

induced by the cosimplicial homomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CA}\backslash B}(H_* S^t \otimes B, G^\bullet C)_\phi &\approx \mathrm{Hom}_{\mathrm{CA}\backslash F_p}(H_* S^t, \mathrm{Map}_{\mathrm{CA}}(B, G^\bullet C)_\phi) \\ &\subset \mathrm{Map}_{\mathrm{CA}}(B, G^\bullet C)_t \end{aligned}$$

§9. An unstable Adams spectral sequence

We now explain how the author and Kan's unstable Adams spectral sequence ([7] and [8]) applies to unpointed mapping spaces, and we briefly discuss the associated homology spectral sequence. This account can easily be adapted to the pointed case. As in Section 8, we let CA be the category of unstable graded commutative coalgebras over the mod-p Steenrod algebra for a fixed prime p, and let $H_* L = H_*(L; F_p)$.

A *mod-p GEM space* is a space whose components are weakly equivalent to products $\Pi_{n=1}^{\infty} K(W_n, n)$ for F_p-modules W_n.

LEMMA 9.1. *There are natural isomorphisms* $[K, L] \approx \mathrm{Hom}_{\mathrm{CA}}(H_*K, H_*L)$ *and*

$$\pi_t(\mathrm{Map}(K, L), f) \approx \mathrm{Hom}_{\mathrm{CA}\backslash H_*K}(H_*S^t \otimes H_*K, H_*L)_{f_*}$$
$$\approx \mathrm{Der}_{\mathrm{CA}}(\tilde{H}_*S^t \otimes H_*K, H_*L)_{f_*}$$

for any space K, *fibrant mod-p GEM space* L, *map* $f: K \to L$, *and* $t \geqq 1$.

PROOF. Using components, we may assume that K and L are connected. Then the first result is well-known and the second follows since $\pi_1(\mathrm{Map}(K, L), f)$ acts trivially on $\pi_0 \mathrm{Fib}_f$, where Fib_f is the fiber of $\mathrm{Map}(S^t \times K, L) \to \mathrm{Map}(K, L)$ over f, because L is equivalent to a simplicial F_p-module and K is a retract of $S^t \times K$.

9.2. The cosimplicial setup. For a space L, let $L \to \bar{F}_p^\bullet L$ be the cosimplicial F_p-resolution given by [8, p. 20]. Then $H_*L \to H_*\bar{F}_p^\bullet L$ is a cosimplicial cofree resolution (8.3) of H_*L in CA. Moreover, the cosimplicial space $\bar{F}_p^\bullet L$ is grouplike [8, p. 276], and thus fibrant, with total space $\mathrm{Tot}\, \bar{F}_p^\bullet L = F_{p\infty}L$ giving the F_p-completion of L, and with tower $\{\mathrm{Tot}_s \bar{F}_p^\bullet L\} = \{F_{ps}L\}$ as in [8, pp. 20–21]. Next for a space K, there is an augmented fibrant cosimplicial space $\mathrm{Map}(K, L) \to \mathrm{Map}(K, \bar{F}_p^\bullet L)$ with total space

$$\mathrm{Tot}\, \mathrm{Map}(K, \bar{F}_p^\bullet L) = \mathrm{Map}(K, F_{p\infty}L)$$

and with tower

$$\{\mathrm{Tot}_s \mathrm{Map}(K, \bar{F}_p^\bullet L)\} = \{\mathrm{Map}(K, F_{ps}L)\}.$$

The connected components (4.4) of $\mathrm{Map}(K, \bar{F}_p^\bullet L)$ correspond to members of

$$\pi^0\pi_0 \mathrm{Map}(K, \bar{F}_p^\bullet L) \approx \pi^0 \mathrm{Hom}_{\mathrm{CA}}(H_*K, H_*\bar{F}_p^\bullet L) \approx \mathrm{Hom}_{\mathrm{CA}}(H_*K, H_*L)$$

and, for each map $\phi: H_*K \to H_*L$ in CA, the connected component $\mathrm{Map}(K, \bar{F}_p^\bullet L)_\phi$ is fibrant with

$$\pi^s\pi_t \mathrm{Map}(K, \bar{F}_p^\bullet L)_\phi \approx \mathrm{Der}^{s,t}_{\mathrm{CA}}(H_*K, H_*L)_\phi$$

for $s \geqq 0$ and $t \geqq 1$ by 9.1 and 8.7.

9.3. The unstable Adams spectral sequence and obstructions. Each map $b: K \to F_{pm}L$ with $1 \leqq m \leqq \infty$ determines a map $b_*: H_*K \to H_*L$ in CA corresponding to $[b] \in \pi^0\pi_0 \mathrm{Map}(K, \bar{F}_p^\bullet L)$ which is given explicitly by the composite of the natural homomorphisms

$$H_*K \xrightarrow{b_*} H_*F_{pm}L \xrightarrow{H_*(\mathrm{proj})} H_*F_{p0} \xrightarrow{H_*(\mathrm{incl})} H_*(F_p \otimes L) \longrightarrow H_*L.$$

When L is F_p-good [8, p. 24], $H_*L = H_*F_{p\infty}L$ and a map $b: K \to F_{p\infty}L$ immediately determines this $b_*: H_*K \to H_*L$. In general, for a map $b: K \to F_{p\infty}L$ we obtain our *unstable Adams spectral sequence*

$$\{E_r^{s,t}(K, L, b)\} = \{E_r^{s,t}(\mathrm{Map}(K, \bar{F}_p^\bullet L), b)\}$$

from 2.5 with

$$E_2^{s,t}(K, L, b) = \mathrm{Der}_{\mathrm{CA}}^{s,t}(H_*K, H_*L)_{b*} \qquad \text{for } s \geqq 0, \quad t \geqq 1$$

$$= \mathrm{Hom}_{\mathrm{CA}}(H_*K, H_*L) \qquad \text{for } s = 0, \quad t = 0$$

by 8.7 and 9.1. More generally, for a map $b: K \to F_{pm}L$ liftable to $(F_p)_{2m+1}L$ with $m \geqq 0$, we obtain a truncated version of this spectral sequence defined for $1 \leqq r \leqq m + 2$. For a given map $\phi: H_*K \to H_*L$ in CA, there always exists a map $b: K \to F_{p1}L$ with $b_* = \phi$ by 5.1, and to realize ϕ we seek a lifting $b: K \to F_{p\infty}L$. By 5.2, for any map $b: K \to F_{pn}L$ with $n \geqq 1$, there are obstructions $\gamma_r(b) \in E_r^{n+1,n}(K, L, b)$ for $1 \leqq r \leqq (n + 3)/2$ which vanish iff the projection of b to $(F_p)_{n-r+1}L$ lifts to $(F_p)_{n+1}L$. In particular, there is an obstruction $\gamma_2(b) \in \mathrm{Der}_{\mathrm{CA}}^{n+1,n}(H_*K, H_*L)_{b*}$. Finally by 5.4, for maps $a, b: K \to F_{p\infty}L$ and a homotopy $H: \Delta^1 \times K \to F_{ps}L$ from a_s to b_s with $s \geqq 0$, there are obstructions

$$\Delta_r(a, H, b) \in E_r^{s+1,s+1}(K, L, b) \qquad \text{for } 1 \leqq r \leqq s + 2$$

which vanish iff the projection of h to $(F_p)_{s-r+1}L$ lifts to a homotopy from a_{s+1} to b_{s+1}. In particular, there is an obstruction $\Delta_2(a, H, b) \in \mathrm{Der}_{\mathrm{CA}}^{s+1,s+1}(H_*K, H_*L)_{b*}$.

9.4. Convergence. For a map $b: K \to F_{p\infty}L$ we have obtained an unstable Adams spectral sequence $\{E_r^{s,t}(K, L, b)\}$ abutting to $\pi_{t-s}(\mathrm{Map}(K, F_{p\infty}L), b)$. If H_*L is of finite type, then this spectral sequence converges completely to $\pi_{t-s}(\mathrm{Map}(K, F_{p\infty}L), b)$ for each $t - s \geqq 0$, and thus $E_{\infty+}^{s,t}(K, L, b) = E_\infty^{s,t}(K, L, b)$ for each $t \geqq s \geqq 0$ and

$$\pi_i(\mathrm{Map}(K, F_{p\infty}L), b) \approx \lim_s \pi_i(\mathrm{Map}(K, F_{ps}L), b).$$

for each $i \geqq 0$. This follows by 4.5 since $\{F_{ps}L\}$, and consequently $\{\mathrm{Map}(K, F_{ps}L)\}$, is weakly equivalent to a continuous tower of simplicial compact Hausdorff spaces which are fibrant as simplicial sets, because the spaces $F_{ps}L$ have finite homotopy.

To construct the corresponding homology spectral sequence, using the homological Lannes functors (8.10), we need

LEMMA 9.5. *There is a natural isomorphism*

$$H_* \operatorname{Map}(K, L) \approx \operatorname{Map}_{CA}(H_*K, H_*L)$$

in CA *for any space K and fibrant mod-p GEM space L.*

PROOF. By 9.1 the natural map from the cofree object $H_* \operatorname{Map}(K, L)$ to the cofree object $\operatorname{Map}_{CA}(H_*K, H_*L)$ in CA induces a bijection

$$\pi_0 H_* \operatorname{Map}(K, L) \approx \pi_0 \operatorname{Map}_{CA}(H_*K, H_*L) \approx \operatorname{Hom}_{CA}(H_*K, H_*L)$$

with π_0 as in 8.1 and an isomorphism

$$\operatorname{Hom}_{CA\backslash F_p}(H_*S^t, H_* \operatorname{Map}(K, L))_\phi \approx \operatorname{Hom}_{CA\backslash F_p}(H_*S^t, \operatorname{Map}_{CA}(H_*K, H_*L))_\phi$$

for each $\phi: H_*K \to H_*L$ in CA and $t \geqq 1$. Thus the natural map is an isomorphism.

9.6. The associated homology spectral sequence. Let

$$\{E_r^{s,t}(K, L; F_p)\} = \{E_r^{s,t}(\operatorname{Map}(K, \bar{F}_p^\bullet L); F_p)\}$$

be the homology spectral sequence of [6] abutting to $H_*(\operatorname{Map}(K, L); F_p)$ and having

$$E_2^{s,t}(K, L; F_p) \approx \operatorname{Map}_{CA}^{s,t}(H_*K, H_*L)$$

by 8.10 and 9.5. For a map $b: K \to F_{p\infty}L$ there is a Hurewicz map of spectral sequences

$$h: \{E_r^{s,t}(K, L, b)\} \to \{E_r^{s,t}(K, L; F_p)\}$$

by 2.7, abutting to

$$h: \pi_{t-s}(\operatorname{Map}(K, L), b) \to H_{t-s}(\operatorname{Map}(K, L); F_p)$$

and given by the homomorphism (8.10)

$$h: \operatorname{Der}_{CA}^{s,t}(H_*K, H_*L)_{b*} \to \operatorname{Map}_{CA}^{s,t}(H_*K, H_*L)$$

when $r = 2$, $s \geqq 0$, and $t \geqq 1$. We show convergence in the standard case.

PROPOSITION 9.7. *If L is an n-connected fibrant space with $n \geqq 1$ and K is a space of dimension $\leqq n$, then $\{E_r^{s,t}(K, L; F_p)\}$ converges strongly to $H_* \operatorname{Map}(K, L) \approx H_* \operatorname{Map}(K, F_{p\infty}L)$.*

PROOF. The natural map

$$\Phi: \{H_i \operatorname{Map}(K, F_{ps}L)\} \to \{H_i T_s(F_p \otimes \operatorname{Map}(K, \bar{F}_p^\bullet L))\}$$

is a pro-isomorphism by [6, 3.2] since $\pi^s\pi_t\,\mathrm{Map}(K, \bar{F}_p^\bullet L) = 0$ for $t \leqq s$ as in [8, p. 31], or alternatively by [6, 3.4] since $\mathrm{Map}^{s,t}_{\mathrm{CA}}(H_*K, H_*L) = 0$ for $t \leqq s$. Thus by [6, 2.3] it will suffice to show that

$$\{H_i\,\mathrm{Map}(K, L)\} \to \{H_i\,\mathrm{Map}(K, F_{p\infty}L)\} \to \{H_i\,\mathrm{Map}(K, F_{ps}L)\}$$

are pro-isomorphisms for each i. When K is a point, this follows by [8, pp. 88, 186]. When K is a (possibly infinite) discrete space, it follows as in [6, 9.3] using the n-connectedness of the spaces $F_{ps}L$ for $0 \leqq s \leqq \infty$ and using the elementary criteria: (i) for a tower $\{A_s \to B_s \to C_s\}$ of fiberings with each B_s and C_s 1-connected, $\{H_iB_s\} \to \{H_iC_s\}$ is a pro-isomorphism for all i iff $\{\tilde{H}_iA_s\}$ is pro-trivial for all i; and (ii) for a tower $\{A_s\}$ of simple spaces, $\{\tilde{H}_iA_s\}$ is pro-trivial for all i iff $\{Z/p \otimes \pi_iA_s\}$ and $\{\mathrm{Tor}(Z/p, \pi_iA_s)\}$ are pro-trivial for all i. When K is a disjoint union $\amalg_\alpha \Delta^m$ of copies of Δ^m for $m \geqq 0$, it follows from the preceding case. In general, it follows by induction on the dimension of K using Eilenberg–Moore spectral sequences for fibre squares of mapping spaces out of

$$\begin{array}{ccc} \coprod_\alpha \dot{\Delta}^m & \longrightarrow & Sk_{m-1}K \\ \big\downarrow \subset & & \big\downarrow \subset \\ \coprod_\alpha \Delta^m & \longrightarrow & Sk_mK \end{array}$$

By Proposition 9.8 below, for $r \geqq 2$ and arbitrary spaces K and L, $\{E_r^{s,t}(K, L; F_p)\}$ is the direct sum of the F_p-homology spectral sequences of the cosimplicial components $\mathrm{Map}(K, \bar{F}_p^\bullet L)_\phi \subset \mathrm{Map}(K, \bar{F}_p^\bullet L)$ for $\phi \in \mathrm{Hom}_{\mathrm{CA}}(H_*K, H_*L)$, and the convergence results of [6] may be applied componentwise.

Proposition 9.8. *For a cosimplicial space $X^\bullet$ and abelian group A, the inclusions of cosimplicial components $X^\bullet_\alpha \subset X^\bullet$ for $\alpha \in \pi^0\pi_0X^\bullet$ induce an isomorphism*

$$\{\oplus_\alpha E_r^{s,t}(X^\bullet_\alpha; A)\} \approx \{E_r^{s,t}(X^\bullet; A)\}$$

for $r \geqq 2$.

This follows from Lemma 9.9 below using $J^\bullet = \pi_0X^\bullet$. A *coefficient system M* on a cosimplicial set $J^\bullet$ consists of a functor M to abelian groups from the category whose objects are the simplices of $J^\bullet$ and whose morphisms $\alpha: x \to y$ are the cosimplicial operators α with $\alpha x = y$. The homomorphism $M(\alpha)$ is written as $\alpha : M_x \to M_y$. We let $H^*(J^\bullet; M) = \pi^*C(J^\bullet; M)$ where $C(J^\bullet; M)$ is the cosimplicial abelian group formed by summing the coefficient groups in each dimension. Using the restriction of M to the constant cosimplicial set $J_c^\bullet$ on $\{v \in J^0 \mid d^0 v = d^1 v\}$, we have

Lemma 9.9. *The inclusion $J_c^\bullet \subset J^\bullet$ induces an isomorphism $H^*(J_c^\bullet; M) \approx H^*(J^\bullet; M)$ for any coefficient system M on a cosimplicial set $J^\bullet$.*

Proof. It suffices to show $H^*(J^\bullet; \bar{M}) = 0$ where $\bar{M}$ is the coefficient system with $\bar{M}_x = 0$ for $x \in J_c^\bullet$ and $\bar{M}_x = M_x$ otherwise. For a simplex $x \in J^\bullet$ and abelian group A, let $L(A, x)$ denote the coefficient system on $J^\bullet$ given by $L(A, x)_y = \bigoplus_I d^I A$ where d^I ranges over the cofacial operators with $d^I x = y$ and where $d^I A = A$. Then $H^*(J^\bullet; L(A, X)) = 0$ since the normalization of $C(J^\bullet; L(A, x))$ is zero except for $\delta : A \approx \delta A$ in dimension $|x|$ and $|x| + 1$. For any coefficient system P on $J^\bullet$, let

$$NP_x = \bigcap_i \ker(s^i : P_x \to P_{s^i x}).$$

Then a homomorphism $P \to P'$ is monic iff $NP_x \to NP'_x$ is monic for all $x \in J^\bullet$. There is a natural homomorphism

$$\bigoplus_{x \in J^\bullet} L(N\bar{M}_x, x) \to \bar{M}$$

which is monic by the above criterion since $N\bar{M}_x = 0$ for $x \in J_c^\bullet$ and $NL(N\bar{M}_x, x)_y = 0$ for $y \neq x$ with $x \notin J_c^\bullet$. The cokernel $\bar{M}'$ of this monomorphism has $H^*(J; \bar{M}) \approx H^*(J; \bar{M}')$, and the lowest nonvanishing group (if any) in $\bar{M}'$ is higher than in $\bar{M}$. Thus, by iteration, $H^*(J; \bar{M}) = 0$.

§10. The E_1-level constructions

We devote the rest of this paper to constructions needed for the results in Sections 2–5, and we start by establishing E_1-level properties of the tower $\{\mathrm{Tot}_m X^\bullet\}$ for a fibrant cosimplicial space $X^\bullet$. Our results here extend those of [8; Ch. X].

10.1. A natural fiber square. Let $M^{m-1}X^\bullet$ be the matching space given by all

$$(x^0, \ldots, x^{m-1}) \in X^{m-1} \times \cdots \times X^{m-1}$$

with $s^i x^j = s^{j-1} x^i$ for $0 \leq i < j \leq m-1$; let $\sigma_m : X^m \to M^{m-1}X^\bullet$ be the fibration with $\sigma_m(x) = (s^0x, \dots, s^{m-1}x)$; and let $\mu_m : \dot{\Delta}^m \subset \Delta^m$. Then there is a natural fiber square

$$\begin{array}{ccc} \mathrm{Tot}_m X^\bullet & \longrightarrow & \mathrm{Map}(\Delta^m, X^m) \\ \big\downarrow & & \big\downarrow \tau_m \\ \mathrm{Tot}_{m-1} X^\bullet & \xrightarrow{\beta_m} & \mathrm{Map}(\mu_m, \sigma_m) \end{array}$$

with the canonical maps. Note that τ_m depends only on the codegeneracies of $X^\bullet$. Clearly, $\mathrm{Map}(\Delta^m, X^m)$ contains X^m as a strong deformation retract via the standard homotopy from the constant map $(d^1)^m(s^0)^m : \Delta^m \to \Delta^m$ to $1 : \Delta^m \to \Delta^m$. For $i, m \geqq 0$ and a vertex $b \in \mathrm{Tot}_m X^\bullet$, there are associated isomorphisms

$$\pi_i \,\mathrm{Fib}_m(X^\bullet, b) \approx \pi_i \,\mathrm{Fib}(\tau_m / b^m) \approx \pi_i \,\mathrm{Fib}(\tau_m / (d^1)^m b_0)$$

where $\mathrm{Fib}_m(X^\bullet, b)$ denotes the fiber of $\mathrm{Tot}_m X^\bullet \to \mathrm{Tot}_{m-1} X^\bullet$ at b and $\mathrm{Fib}(\tau_m / x)$ denotes the fiber of τ_m at a vertex x. For any vertex $v \in X^m \subset \mathrm{Map}(\Delta^m, X^m)$, e.g. $v = (d^1)^m b_0$, there is a canonical isomorphism

$$\mathrm{Fib}(\tau_m / v) \approx \mathrm{Map}_*(S^m, \mathrm{Fib}(\sigma_m / v))$$

where $\mathrm{Fib}(\sigma_m / v)$ is the fibre of $\sigma_m : X^m \to M^{m-1}X$ at v, and this induces an isomorphism $\pi_i \,\mathrm{Fib}(\tau_m / v) \approx \pi_{i+m} \,\mathrm{Fib}(\sigma_m / v)$ using the standard orientation of $S^i \wedge S^m$. By [6, §5] there is a canonical isomorphism

$$\pi_{i+m} \,\mathrm{Fib}(\sigma_m / v) \approx N\pi_{i+m}(X^m, v) \qquad \text{when } m = 0 \quad \text{or} \quad v \in X_c^m,$$

where $X_c^\bullet$ is the union of the connected components of $X^\bullet$ (see 4.4). The above isomorphisms compose to give the following Φ.

Proposition 10.2. *For $i, m \geqq 0$ and a vertex $b \in \mathrm{Tot}_m X^\bullet$, there is a natural isomorphism*

$$\Phi : \pi_i \,\mathrm{Fib}_m(X^\bullet, b) \approx N\pi_{i+m}(X^m, b)$$

of groups when $i \geqq 1$ and of pointed sets when $i = 0$.

In the fiber square 10.1, we next determine $\pi_*(\mathrm{Map}(\mu_m, \sigma_m), v)$ for $m \geqq 1$ and $v \in X_c^m$. The map τ_m induces

$$\tau_{m*} : \pi_i(X^m, v) \to \pi_i(\mathrm{Map}(\mu_m, \sigma_m), v)$$

and, for the pointed space $(\dot{\Delta}^m, 0)$, the inclusion

$$\theta_m : \mathrm{Map}_*(\mathring{\Delta}^m, \mathrm{Fib}(\sigma_m, v)) \subset \mathrm{Map}(\mu_m, \sigma_m)$$

induces

$$N\pi_{m+i-1}(X^m, v) \approx \pi_{m+i-1}\,\mathrm{Fib}(\sigma_m/v)$$
$$\approx \pi_i\,\mathrm{Map}_*(\mathring{\Delta}^m, \mathrm{Fib}(\sigma_m/v)) \xrightarrow{\theta_{m*}} \pi_i(\mathrm{Map}(\mu_m, \sigma_m), v)$$

using the standard orientation of $S^i \wedge \mathring{\Delta}^m$. The right action of $\pi_1(X^m, v)$ on $N\pi_*(X^m, v) \subset \pi_*(X^m, v)$ gives a semidirect product $\pi_1(X^m, v) \times_\phi N\pi_m(X^m, v)$, which is defined as the set $\pi_1(X^m, v) \times N\pi_m(X^m, v)$ with multiplication $(g, a)\cdot(h, b) = (gh, ah + b)$.

PROPOSITION 10.3. *For $m \geqq 1$ and $v \in X^m_c$, there are natural isomorphisms*

$$\tau_{m*}\theta_{m*} : \pi_1(X^m, v) \times_\phi N\pi_m(X^m, v) \approx \pi_1(\mathrm{Map}(\mu_m, \sigma_m), v),$$
$$\tau_{m*} + \theta_{m*} : \pi_i(X^m, v) \oplus N\pi_{m+i-1}(X^m, v) \approx \pi_i(\mathrm{Map}(\mu_m, \sigma_m), v),$$

for $i \geqq 2$. The composite of these isomorphisms with

$$\partial_* : \pi_i(\mathrm{Map}(\mu_m, \sigma_m), v) \to \pi_{i-1}\,\mathrm{Fib}(\tau_m/v) \approx N\pi_{i+m-1}(X^m, v)$$

carries each element (a, b) to $(1)^{i-1}b$ for $i \geqq 1$.

PROOF. The results on $\pi_i(\mathrm{Map}(\mu_m, \sigma_m), v)$ follow since τ_m restricts to a cross-section of the fibration $\varepsilon_m : \mathrm{Map}(\mu_m, \sigma_m) \to X^m$ with $\varepsilon_m(f) = f(0)$ for $0 \in \mathring{\Delta}^m$ and since θ_m makes $\mathrm{Map}_*(\mathring{\Delta}^m, \mathrm{Fib}(\sigma_m/v))$ a homotopy fiber of ε_m. The result on ∂_* follows using the map of fiber sequences

$$\begin{array}{ccccc} \mathrm{Map}_*(S^m, \mathrm{Fib}(\sigma_m/v)) & \to & \mathrm{Map}_*(\Delta^m, \mathrm{Fib}(\sigma_m/v)) & \to & \mathrm{Map}_*(\mathring{\Delta}^m, \mathrm{Fib}(\sigma_m/v)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_*(S^m, \mathrm{Fib}(\sigma_m/v)) & \to & \mathrm{Map}(\Delta^m, X^m) & \to & \mathrm{Map}(\mu_m, \sigma_m). \end{array}$$

Using the operator δ of 2.2, we have

PROPOSITION 10.4. *For $i, m \geqq 1$ and a vertex $b \in \mathrm{Tot}_m X^\bullet$, the fibration composite*

$$\pi_i\,\mathrm{Fib}_{m-1}(X^\bullet, b) \longrightarrow \pi_i(\mathrm{Tot}_{m-1} X^\bullet, b) \xrightarrow{\partial} \pi_{i-1}\,\mathrm{Fib}_m(X^\bullet, b)$$

corresponds to $(-1)^{i-1}\delta : N\pi_{i+m-1}(X^{m-1}, b) \to N\pi_{i+m-1}(X^m, b)$.

PROOF. Taking a fibration mapping cone of $b : Sk_m\mathring{\Delta} \to X^\bullet$, we may assume by naturality that $X^\bullet$ is pointed and that $b \in \mathrm{Tot}_m X^\bullet$ is at the basepoint. The result now follows using 10.3 since the composite of canonical maps

$$\mathrm{Fib}_{m-1} X^\bullet \to \mathrm{Tot}_{m-1} X^\bullet \to \mathrm{Map}(\mu_m, \sigma_m) \to \mathrm{Map}(\dot{\Delta}^m, X^m)$$

equals the composite

$$\mathrm{Map}_*(S^{m-1}, \mathrm{Fib}(\sigma_m)) \to \mathrm{Map}_*(S^{m-1} \vee \cdots \vee S^{m-1}, X^m) \to \mathrm{Map}(\dot{\Delta}^m, X^m)$$

of maps induces by $d^i : \mathrm{Fib}(\sigma_{m-1}) \to X^m$ for $0 \leqq i \leqq m$ and by the skeletal quotient map $\dot{\Delta}^m \to S^{m-1} \vee \cdots \vee S^{m-1}$.

By the naturality of Φ in b, the fibration right action of $\pi_1(\mathrm{Tot}_m X^\bullet, b)$ on $\pi_i \mathrm{Fib}_m(X^\bullet, b) \approx N\pi_{i+m}(X^m, b)$ agrees with the fundamental action (3.2). For $i = 0$ we have more generally

Proposition 10.5. *For a vertex $b \in \mathrm{Tot}_m X^\bullet$ with $m \geqq 1$, the fibration boundary*

$$\Phi\partial_* : \pi_1(\mathrm{Tot}_{m-1} X^\bullet, b) \to \pi_0 \mathrm{Fib}_m(X^\bullet, b) \approx N\pi_m(X^m, b)$$

is a crossed-homomorphism with respect to the fundamental action of $\pi_1(\mathrm{Tot}_{m-1} X^\bullet, b)$ on $N\pi_m(X^m, b)$, and the associated crossed-homomorphism action (2.2) *agrees with the fibration action.*

Proposition 10.5. *For a vertex $b \in \mathrm{Tot}_m X^\bullet$ with $m \geqq 1$, the fibration boundary*

$$\Phi\partial_* : \pi_1(\mathrm{Tot}_{m-1} X^\bullet, b) \to \pi_0 \mathrm{Fib}_m(X^\bullet, b) \approx N\pi_m(X^m, b)$$

is a crossed-homomorphism with respect to the fundamental action of $\pi_1(\mathrm{Tot}_{m-1} X^\bullet, b)$ on $N\pi_m(X^m, b)$, and the associated crossed-homomorphism action (2.2) *agrees with the fibration action.*

Proof. By 10.1–10.3, $\Phi\partial_*$ is the composite of the homomorphism

$$\beta_{m*} : \pi_1(\mathrm{Tot}_{m-1} X^\bullet, b) \to \pi_1(X^m, b) \times_\phi N\pi_m(X^m, b)$$

with the projection function to $N\pi_m(X^m, b)$, and thus $\Phi\partial_*$ is a crossed-homomorphism. The fibration action of $\pi_1(\mathrm{Tot}_{m-1} X^\bullet, b)$ is determined via the fibration τ_m.

For $m \geqq 1$ consider the commutative triangle

$$\begin{array}{ccc} \pi_0 \operatorname{Tot}_{m-1} X^\bullet & \xrightarrow{\beta_{m*}} & \pi_0 \operatorname{Map}(\mu_m, \sigma_m) \\ & \overset{\bar{c}}{\searrow} & \downarrow \alpha_m \\ & & N\pi^{\text{free}}_{m-1} X^m \end{array}$$

using β_m from 10.1 and the canonical maps to $N[\dot{\Delta}^m, X^m]_{\text{free}}$ which is here identified with $N\pi^{\text{free}}_{m-1} X^m$ via the reverse orientation (5.1) of $\dot{\Delta}^m$. The proof of 10.3 shows

PROPOSITION 10.6. *If $X^\bullet_c = X^\bullet$ and $m \geqq 1$, then there is a natural bijection*

$$\alpha_m : \pi_0 \operatorname{Map}(\mu_m, \sigma_m) \approx N\pi^{\text{free}}_{m-1} X^m.$$

Our next result leads to the obstruction cocycle of 5.1.

PROPOSITION 10.7. *For $m \geqq 1$, a vertex $a \in \operatorname{Tot}_{m-1} X^\bullet$ lifts to $\operatorname{Tot}_m X^\bullet$ if and only if $a^m : \dot{\Delta}^m \to X^m$ is nullhomotopic.*

PROOF. Using 4.4 we may assume that $X^\bullet_c = X^\bullet$. Then 10.1 and 10.6 show that $[a] \in \pi_0 \operatorname{Tot}_{m-1} X^\bullet$ lifts to $\pi_0 \operatorname{Tot}_m X^\bullet$ iff the element $\bar{c}[a] = [a^m]$ is trivial in $N\pi^{\text{free}}_{m-1} X^m$.

Finally we consider

10.8. Hurewicz maps. The R-homology spectral sequence of $X^\bullet$ is constructed in [6] by using the total chain complex $T(R \otimes X^\bullet)$ with

$$T(R \otimes X^\bullet)_n = \prod_{m \geqq 0} N^m N_{m+n}(R \otimes X^\bullet),$$

$$\partial_T = \partial + (-1)^{n+1}\delta : T(R \otimes X^\bullet)_n \to T(R \otimes X^\bullet)_{n-1},$$

where $N^* N_*(R \otimes X^\bullet)$ is the normalized double complex with $\partial = \Sigma_i (-1)^i d_i$ and $\delta = \Sigma_i (-1)^i d^i$. This is filtered by the subcomplexes $F^m T(R \otimes X^\bullet)$ with

$$F^m T(R \otimes X^\bullet)_n = \prod_{k \geqq m} N^k N_{k+n}(R \otimes X^\bullet)$$

and there is an associated tower of complexes

$$T_m(R \otimes X^\bullet) = T(R \otimes X^\bullet)/F^{m+1} T(R \otimes X^\bullet)$$

producing the homology spectral sequence with E_1-terms

$$H_i(F^m T(R \otimes X^\bullet)/F^{m+1} T(R \otimes X^\bullet)) \approx NH_{i+m}(X^m; R).$$

The natural chain map

$$N_*(R\otimes \mathrm{Tot}_m X^\bullet)\subset N_*(\mathrm{Tot}_m(R\otimes X^\bullet)) \xrightarrow{\phi} T_m(R\otimes X^\bullet)$$

of [6, 2.2] induces a map

$$\phi_*: H_*(\mathrm{Tot}_m X^\bullet; R)\to H_*T_m(R\otimes X^\bullet).$$

Using the map h of 2.7 we obtain a commutative ladder of exact sequences

$$\begin{array}{ccccccc}
\cdots\to\pi_{i+1}(\mathrm{Tot}_{m-1}X^\bullet, b) & \to & N\pi_{i+m}(X^m, b) & \to & \pi_i(\mathrm{Tot}_m X^\bullet, b)\to\cdots \\
\downarrow{\scriptstyle \phi_* h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \phi_* h} \\
\cdots\to H_{i+1}T_{m-1}(R\otimes X^\bullet) & \to & NH_{i+m}(X^m; R) & \to & H_iT_m(R\otimes X^\bullet)\to\cdots
\end{array}$$

terminating with $\pi_0(\mathrm{Tot}_{m-1}X^\bullet, b)$ and $H_0T_{m-1}(R\otimes X^\bullet)$. For vertices $a, b\in \mathrm{Tot}_m X^\bullet$ with $a_{m-1}=b_{m-1}\in\mathrm{Tot}_{m-1}X^\bullet$, the obstruction cocycle and difference cochain of 5.1 and 5.3 satisfy

$$hc(b)=\partial_{T*}[\phi b]\in NH_m(X^{m+1}; R),$$

$$hD(a, b)=[\phi a-\phi b]\in NH_m(X^m; R).$$

Moreover, the diagram

$$\begin{array}{ccc}
\pi_0\,\mathrm{Tot}_{m-1}X^\bullet & \xrightarrow{c} & N\pi^{\mathrm{free}}_{m-1}X^m \\
\downarrow{\scriptstyle \phi_* e} & & \downarrow{\scriptstyle h} \\
H_0T_{m-1}(R\otimes X_\bullet) & \xrightarrow{\partial_{T*}} & NH_{m-1}(X_m; R)
\end{array}$$

commutes where, for any space Y, $e:\pi_0 Y\to H_0(Y; R)$ is defined by $e[y]=[y]$.

§11. Construction of homotopy differentials: Part I

For a fibrant cosimplicial space $X^\bullet$, we shall construct relations (see 2.3)

$$d_r: N\pi_q(X^m, b)\rightharpoonup N\pi_{q+r-1}(X^{m+r}, b),$$

called *differentials*, which will induce the required spectral sequence differentials. We assume $q\geqq m$ here and postpone the case $q<m$ to Section 12. Using the isomorphism Φ of 10.2 and the obstruction $c(\ \ )$ and $D(\ \ ,\ \ )$ of 5.1 and 5.3, we first specify:

11.1. Ordinary differentials. For $r\geqq 1$, $q>m\geqq 0$, and a vertex $b\in \mathrm{Tot}_{m+r}X^\bullet$, the *ordinary differential*

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$$

is defined as the composite relation of

$$N\pi_q(X^m, b) \overset{\Phi}{\approx} \pi_{q-m}\,\mathrm{Fib}_m(X^\bullet, b) \longrightarrow \pi_{q-m}(\mathrm{Tot}_m X^\bullet, b).$$

$$\longleftarrow \pi_{q-m}(\mathrm{Tot}_{m+r-1} X^\bullet, b) \xrightarrow{\partial_*} \pi_{q-m-1}\,\mathrm{Fib}_{m+r}(X^\bullet, b) \overset{\Phi}{\approx} N_{q+r-1}(X^{m+r}, b),$$

For $r, m \geqq 1$ and a vertex $b \in \mathrm{Tot}_{m+r-1} X^\bullet$, the *ordinary differential*

$$d_r: N\pi_m(X^m, b) \to N\pi_{m+r-1}(X^{m+r}, b)$$

is defined by letting $d_r(\theta) = \phi$ iff there exists a vertex $a \in \mathrm{Tot}_{m+r-1} X^\bullet$ such that $a_{m-1} = b_{m-1}$, $D(a_m, b_m) = \theta$, and $c(a) - c(b) = \phi$. When $X^\bullet$ is pointed and b is at the basepoint, the ordinary differentials will be called *pointed differentials*.

LEMMA 11.2. *Suppose that the ordinary differential d_r is defined on $N\pi_q(X^m, q)$, and let $f: X^\bullet \to Y^\bullet$ be a map to a fibrant cosimplicial space $Y^\bullet$ such that $f_*: N\pi_j(X^k, b) \approx N\pi_j(Y^k, fb)$ for $m \leqq k \leqq m + r$, $q \leqq j \leqq q + r - 1$, and $j - k \geqq -1$. Then the ordinary differentials*

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b),$$

$$d_r: N\pi_q(Y^m, fb) \to N\pi_{q+r-1}(Y^{m+r}, fb)$$

correspond under f_*.

PROOF. For the case $q = m \geqq 1$, let $F_k X^\bullet$ be the fibre of $\mathrm{Tot}_k X^\bullet \to \mathrm{Tot}_{m-1} X^\bullet$ over $b \in \mathrm{Tot}_{m-1} X^\bullet$. Then by induction on k, $f_*: \pi_t(F_k X^\bullet, b) \approx \pi_t(F_k Y^\bullet, fv)$ for each vertex $v \in F_k X^\bullet$, $m \leqq k \leqq m + r - 1$ and $0 \leqq t \leqq m + r - 1 - k$. The result follows since d_r is the composite of

$$N\pi_m(X^m, b) \overset{\Phi}{\approx} \pi_0(F_m X^\bullet, b) \leftarrow \pi_0(F^{m+r-1} X^\bullet, b) \xrightarrow{\gamma} N\pi_{m+r-1}(X^{m+r}, b)$$

with $\gamma[a] = c(a) - c(b)$. The result for $q > m \geqq 0$ follows similarly.

11.3. General differentials. These are relations $d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$ which will be defined for $r, q \geqq 1$ and $q \geqq m \geqq 0$, where $b \in \mathrm{Tot}_r X^\bullet$ is a vertex such that the Whitehead products between $\pi_q(X^{m+r}, b)$ and the image of $b_*: \pi_r(Sk_r\Delta^{m+r}, 0) \to \pi_r(X^{m+r}, b)$ are trivial in $\pi_{q+r-1}(X^{m+r}, b)$. This condition holds automatically when b lifts to $\mathrm{Tot}_{r+1} X^\bullet$ or when X^{m+r} has trivial Whitehead products or when $m = 0$. First take the pull-back

$$\begin{array}{ccc} \bar{X}^\bullet & \xrightarrow{u} & X^\bullet \\ \Big\downarrow{\scriptstyle v} & & \Big\downarrow \\ Sk_r\Delta^\bullet & \xrightarrow{b} & P_{q-1}X^\bullet \end{array}$$

of cosimplicial spaces where $P_{q-1}X^\bullet$ the $(q-1)$st Postinikov section [18, p. 32] of $X^\bullet$, and regard the obvious cross-section $b: Sk_r\Delta^\bullet \to \bar{X}^\bullet$ as an inclusion. Then the cosimplicial maps $u: \bar{X}^\bullet \to X^\bullet$ and $j: \bar{X}^\bullet \to (\bar{X}/Sk_r\Delta)^\bullet$ induce isomorphisms

$$\pi_t(X^k, b) \approx \tilde{\pi}_t(\bar{X}^k, b) \approx \pi_t(\bar{X}/Sk_r\Delta)^k$$

for $q \leqq t \leqq q+r-1$ and $k \leqq m+r$, where $\tilde{\pi}_t(\bar{X}^k, b)$ denotes the kernel of $v_*: \pi_t(\bar{X}^k, b) \to \pi_t(Sk_r\Delta^k, 0)$. Using the associated isomorphisms $N\pi_t(X^k, b) \approx N\pi_t(\bar{X}/Sk_r\Delta)^k$ for $q \leqq t \leqq q+r-1$ and $k \leqq m+r$, we let the *general differential*

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$$

correspond to the pointed differential

$$d_r: N\pi_q(\bar{X}/Sk_r\Delta)^m \to N\pi_{q+r-1}(\bar{X}/Sk_r\Delta)^{m+r}$$

given by 11.1 after making $(\bar{X}/Sk_r\Delta)^\bullet$ fibrant.

11.4. Improved general differentials. These are relations

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$$

which will be defined for $r, q \geqq 1$ and $q \geqq m \geqq 0$ where $b \in \mathrm{Tot}_{r-1} X^\bullet$ is a vertex liftable to $\mathrm{Tot}_r X^\bullet$ and such that

$$[\pi_r(X^{m+r}, b), \pi_q(X^{m+r}, b)] = 0 \qquad \text{in } \pi_{r+q-1}(X^{m+r}, b).$$

These d_r are given by the corresponding general differentials of 11.3 using an arbitrary lifting of b to $\mathrm{Tot}_r X^\bullet$. Two liftings $b', b'' \in \mathrm{Tot}_r X^\bullet$ give the same relation by a straightforward argument using the map

$$b' \amalg b'': Sk_r\Delta^\bullet \coprod_{Sk_{r-1}\Delta^\bullet} Sk_r\Delta^\bullet \to P_{q-1}X^\bullet$$

in place of $b: Sk_r\Delta^\bullet \to P_{q-1}X^\bullet$ in 11.3.

An ordinary $d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$ may be compared with the associated general d_r on $N\pi_q(X^m, b)$ when the latter is also defined, i.e. when $(q, m) \neq (1, 1)$ or when $(q, m) = (1, 1)$ with $\pi_1(X^{r+1}, b)$ acting trivially on $c(b) \in N\pi_r(X^{r+1}, b)$ (as it automatically does when b lifts to $\mathrm{Tot}_{r+1} X^\bullet$).

THEOREM 11.5. *An ordinary differential*

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$$

equals the associated general differential when the latter is defined.

PROOF. First suppose that $q > m$ and $b \in \mathrm{Tot}_{m+r} X^\bullet$. Letting $\bar{X}'^\bullet$ be the pullback of $b: Sk_{m+r}\Delta^\bullet \to P_{q-1}X^\bullet \leftarrow X^\bullet$ and using the cosimplicial maps

$$X^\bullet \leftarrow \bar{X}'^\bullet \to (\bar{X}'/Sk_{m+r}\Delta)^\bullet \leftarrow (\bar{X}/Sk_r\Delta)^\bullet$$

with objects made fibrant, Lemma 11.2 shows that the ordinary differentials d_r agree on the successive groups

$$N\pi_q(X^m, b) \approx N\pi_q(\bar{X}'^m, b) \approx N\pi_q(\bar{X}'/Sk_{m+r}\Delta)^m \approx N\pi_q(\bar{X}/Sk_r\Delta)^m.$$

The result now follows since the ordinary d_r on $N\pi_q(\bar{X}/Sk_r\Delta)^m$ agrees by definition with the general d_r or $N\pi_q(X^\bullet, b)$. Next suppose that $q = m$ and $b \in \mathrm{Tot}_{m+r-1} X^\bullet$. It suffices to show that the ordinary d_r on $N\pi_m(X^m, b)$ corresponds to the pointed d_r on $N\pi_m(\bar{X}'Sk_{m+r-1}\Delta)^m$ where $\bar{X}'^\bullet$ is now the pullback of

$$b: Sk_{m+r-1}\Delta^\bullet \to P_{m-1}X^\bullet \leftarrow X^\bullet.$$

For an ordinary $d_r\theta = \phi$ determined by an element $a \in \mathrm{Tot}_{m+r-1} X^\bullet$ with $a_{m-1} = b_{m-1}$, there is a corresponding pointed differential constructed using the cross-section $a: Sk_{m+r-1}\Delta^\bullet \to \bar{X}'^\bullet$. The converse follows by naturality after first applying Lemma 11.2 to the cosimplicial map

$$\bar{X}'^\bullet \to (\bar{X}'/Sk_{m+r-1}\Delta)^\bullet \times Sk_{m+r-1}\Delta^\bullet$$

with objects made fibrant.

COROLLARY 11.6. *Let $a, a', b \in \mathrm{Tot}_{m+r-1} X^\bullet$ be vertices such that $a'_{m-1} = a_{m-1}$ in $\mathrm{Tot}_{m-1} X^\bullet$. Suppose that $1 \leqq r < m$ and $a_r = b_r$, or that $1 \leqq r \leqq m$ and $a_{r-1} = b_{r-1}$ with $[\pi_r(X^{m+r}, a), \pi_m(X^{m+r}, a)] = 0$. Then there exists a vertex $b' \in \mathrm{Tot}_{m+r-1} X^\bullet$ such that $b'_{m-1} = b_{m-1}$ in $\mathrm{Tot}_{m-1} X^\bullet$, $D(b'_m, b_m) = D(a'_m, a_m)$, and $c(b') - c(b) = c(a') - c(a)$.*

Since general differentials are defined in terms of pointed differentials, the following properties are easily verified by reducing to the pointed case.

11.7. Formula for d_1. A general $d_1: N\pi_q(X^m, b) \to N\pi_q(X^{m+1}, b)$ equals the function $(-1)^{q-m-1}\delta$ of 2.2 and 10.4.

11.8. Naturality properties. A general $d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$

is natural in $X^\bullet$ and $b \in \mathrm{Tot}_r X^\bullet$ (or $b \in \mathrm{Tot}_{r-1} X^\bullet$ in the improved case 11.4), i.e. for cosimplicial maps and path classes.

11.9. Additivity properties. For additive (but possibly non-abelian) groups A and B, a relation $f: A \rightharpoonup B$ is called *additive* if: (i) $h(0)=0$; (ii) $h(a)=b$ implies $h(-a)=-b$; and (iii) $h(a_1)=b_1$ and $h(a_2)=b_2$ imply $h(a_1+a_2)=b_1+b_2$. For a group G right-acting on B, a relation $k: G \rightharpoonup B$ is called *crossed-additive* if: (i) $k(e)=0$; (ii) $k(g)=b$ implies $k(g^{-1})=-bg^{-1}$; and (iii) $k(g_1)=b_1$ and $k(g_2)=b_2$ imply $k(g_1g_2)=b_1g_2+b_2$. A general $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ always gives $d_r(0)=0$ and is additive for $q>m$ with $(m,q)\neq(0,1)$ or $q=m>r$ (or $q=m=r$ in the improved case 11.4). A general $d_r: N\pi_1(X^0, b) \rightharpoonup N\pi_r(X^r, b)$ is crossed-additive using the fundamental action (3.2) of $\pi_1(X^0, b)$ on $N\pi_r(X^r, b)$. These results follow by applying 10.5 and 11.6 after reducing to the pointed case.

11.10. Domain and indeterminacy properties. For a general $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ with $r \geqq 2$, the domain of d_r equals the kernel of d_{r-1} and the indeterminacy of d_r equals the image of d_{r-1}. For a general $d_r: N\pi_m(X^m, b) \rightharpoonup N\pi_{m+r-1}(X^{m+r}, b)$ there are further indeterminacy properties in cases where d_r may be non-additive. If $d_r x = y$ with $r \geqq m$, then $d_r x = y + z$ for each element $z \in N\pi_{m+r-1}(X^{m+r}, b)$ in the indeterminacy of d_m. If $d_r x = y$ and $d_r x = y'$ with $r = m$, then the element $y - y'$ lies in the indeterminacy of d_r. Furthermore, in the improved case (i.e. when $[\pi_r(X^{m+r}, b), \pi_m(X^{m+r}, b)] = 0$), if $d_r x = y$ with $r \geqq m+1$, then $d_r x = y + z$ for each element $z \in N\pi_{m+r-1}(X^{m+r}, b)$ in the indeterminacy of d_{m+1}. Finally, in the improved case, if $d_r x = y$ and $d_r x = y'$ with $r = m+1$, then the element $y' - y$ lies in the indeterminacy of d_r. These results follow by reducing to the pointed case and using 11.6 when needed.

11.11. Composition properties. For $q, r, s \geqq 1$ and $q > m \geqq 0$, let $b \in \mathrm{Tot}_{r+s-1} X^\bullet$ be a vertex with trivial Whitehead products between $\pi_q(X^{m+r+s}, b)$ and the image of

$$b_*: \pi_{r+s-1}(Sk_{r+s-1}\Delta^{m+r+s}, 0) \to \pi_{r+s-1}(X^{m+r+s}, b).$$

This condition on $b \in \mathrm{Tot}_{r+s-1} X^\bullet$ holds automatically when b lifts to $\mathrm{Tot}_{r+s} X^\bullet$ or when X^{m+r+s} has trivial Whitehead products. The general differentials

$$N\pi_q(X^m, b) \xrightarrow{d_r} N\pi_{q+r-1}(X^{m+r}, b) \xrightarrow{d_s} N\pi_{q+r+s-2}(X^{m+r+s}, b)$$

are now defined and have range $d_r \subset \ker d_s$. Moreover, if $d_r(w) = x$ and $d_s(y) = z$, then $d_s(y + x) = z$ when $(m, q) \neq (0, 1)$ and $d_s(yw + x) = zw$ when $(m, q) = (0, 1)$ with yw given by the fundamental action (3.2). These results follow by reducing to the pointed case of $(\bar{X}/Sk_{r+s-1}\Delta)^\bullet$ made fibrant, where $\bar{X}^\bullet$ is the pullback of $b: Sk_{r+s-1}\Delta^\bullet \to P_{q-1}X^\bullet \leftarrow X^\bullet$ as in 11.3.

11.12. Equivariance properties. Suppose that the general differentials $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ and $d_r: N\pi_1(X^0, b) \rightharpoonup N\pi_r(X^r, b)$ respectively give $d_r(y) = z$ and $d_r(w) = 0$. Then the former gives $d_r(yw) = zw$ using the fundamental action of $\pi_1(X^0, b)$. This follows by the naturality of d_r in b, since w must lift to $\pi_1(\mathrm{Tot}_r X^\bullet, b)$.

11.13. Hurewicz properties. For a general differential $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ and ring R with identity, there is a corresponding homology differential $d_r: NH_q(X^m; R) \rightharpoonup NH_{q+r-1}(X^{m+r}; R)$ of [6] which gives $d_r(hx) = hy$ whenever $d_r x = y$ for $x \in N\pi_q(X^m, b)$. This follows using 10.8.

Finally, we introduce

11.14. The bottom differential. For $r \geqq 1$, the *bottom differential* $d_r: \pi_0 X^0 \rightharpoonup N\pi_{r-1}^{\mathrm{free}} X^r$ is defined as the composite of the relations

$$\pi_0 X^0 = \pi_0 \mathrm{Tot}_0 X^\bullet \leftarrow \pi_0 \mathrm{Tot}_{r-1} X^\bullet \xrightarrow{\bar{c}} N\pi_{r-1}^{\mathrm{free}} X^r$$

with $\bar{c}$ as in 10.6. The *free kernel of* d_r is defined as the set of all $x \in \pi_0 X^0$ such that $d_r x = p$ for some trivial $p \in N\pi_{r-1}^{\mathrm{free}} X^r$. By 10.7, this free kernel equals the image of $\pi_0 \mathrm{Tot}_r X^\bullet \to \pi_0 X^0$, which equals the domain of d_{r+1}. For a ring R with identity, there is a corresponding homology differential $d_r: H_0(X^0; R) \rightharpoonup NH_{r-1}(X^r; R)$ which gives $d_r(ex) = hy$ by 10.8 whenever the bottom differential gives $d_r x = y$ for $x \in \pi_0 X^0$.

§12. Construction of homotopy differentials: Part II

For a fibrant cosimplicial space $X^\bullet$, we now construct differentials $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ in the case $q < m$. As in [9], we use universal examples in the homotopy category $\mathrm{Ho}(\nabla S_*)$ of pointed cosimplicial spaces (see [8, p. 277]).

DEFINITION 12.1. For $1 < q < m$ and $1 \leqq r \leqq q$, a *$d_r^{m,q}$-model* is a pointed cosimplicial space $M^\bullet$ with elements $i \in N\pi_q M^m$ and $j \in N\pi_{q+r-1} M^{m+r}$ such that:

(i) M^s is simply connected for $s \geqq 0$.

(ii) The integral homology spectral sequence [6] of $M^\bullet$ has $d^r: NH_qM^m \approx NH_{q+r-1}M^{m+r} \approx Z$, $NH_0M^0 \approx Z$, and $NH_tM^s \approx 0$ otherwise.

(iii) The elements $h(i) \in NH_qM^m$ and $h(j) \in NH_{q+r-1}M^{m+r}$ are generators with $d_rh(i) = h(j)$.

LEMMA 12.2. *Let $M^\bullet$ be a $d_r^{m,q}$-model for $1 < q < m$ and $1 \leqq r \leqq q$. Then for $t < 2q - 1$, the group $N\pi_tM^s$ is isomorphic to π_tS^q when $s = m$, to π_tS^{q+r-1} when $s = m + r$, and to 0 otherwise. The group $N\pi_{2q-1}M^s/N[\pi_qM^s, \pi_qM^s]$ is isomorphic to $\pi_{2q-1}S^q/[\pi_qS^q, \pi_qS^q]$ when $s = m$, to $\pi_{2q-1}S^{q+r-1}/[\pi_qS^{q+r-1}, \pi_qS^{q+r-1}]$ when $s = m + r$, and to 0 otherwise.*

PROOF. By 12.1, $h: \pi_tM^s \to H_tM^s$ is onto for $t \geqq 1$ and $s \geqq 0$, and thus each M^s is weakly equivalent to a wedge of q-spheres and $(q + r - 1)$-spheres. Let $F\pi_tM^s \subset \pi_tM^s$ denote the subgroup of elements in the image of $f_*: \pi_t(S^q \vee \cdots \vee S^q) \to \pi_tM^s$ for some map $f: S^q \vee \cdots \vee S^q \to M^s$. There are cosimplicial isomorphisms

$$F\pi_tM^\bullet \approx \pi_tS^q \otimes H_qM^\bullet \qquad \text{for } t < 2q - 1,$$

$$\pi_tM^\bullet/F\pi_tM^\bullet \approx \pi_tS^{q+r-1} \otimes H_{q+r-1}M^\bullet \qquad \text{for } t < 2q - 1,$$

$$F\pi_{2q-1}M^\bullet/[\pi_qM^\bullet, \pi_qM^\bullet] \approx (\pi_{2q-1}S^q/[\pi_qS^q, \pi_qS^q]) \otimes H_qM^\bullet,$$

$$\pi_{2q-1}M^\bullet/F\pi_{2q-1}M^\bullet \approx \begin{cases} \pi_{2q-1}S^{q+r-1} \otimes H_{q+r-1}M^\bullet & \text{for } r > 1, \\ 0 & \text{for } r = 1, \end{cases}$$

and the lemma follows since normalization is exact and commutes with additive functors.

12.3. The models $D_r^{m,q}$. For $0 < q < m$ let $M_1^{m,q}$ be the pointed cosimplicial space such that $(M_1^{m,q})^s$ equals: $*$ for $s < m$; S^q for $s = m$; and the wedge $\bigvee_I d^IS^q$ for $s > m$, where $d^IS^q = S^q$ with d^I ranging over the cofacial operators from dimension m to s. Choose a weak equivalence $M_1^{m,q} \to D_1^{m,q}$ to a fibrant-cofibrant pointed cosimplicial space $D_1^{m,q}$. Let $i \in N\pi_q(D_1^{m,q})^m$ be represented by $S^q \subset (M_1^{m,q})^q$ and let $j \in N\pi_q(D_1^{m,q})^{m+1}$ be $j = (-1)^{q-m+1}\delta(i)$ with δ as in 2.2. Then $D_1^{m,q}$ is a $d_1^{m,q}$-model when $q > 1$. Moreover, for each pointed cosimplicial space $Y^\bullet$, there is a bijection $[D_1^{m,q}, Y^\bullet] \approx N\pi_qY^m$ sending each f to $f_*(i)$. Proceeding inductively, given the $d_{r-1}^{m,q}$-model $D_{r-1}^{m,q}$ with $2 \leqq r \leqq q$, we then construct $D_r^{m,q}$ by forming the homotopy cofibering

$$D_1^{m+r-1,q+r-2} \xrightarrow{j} D_{r-1}^{m,q} \xrightarrow{k} D_r^{m,q}$$

with $D_r^{m,q}$ fibrant–cofibrant and with the map j representing $j \in N\pi_{q+r-2}(D_{r-1}^{m,q})^{m+r-1}$. Using Mayer–Vietoris sequences for the homology theories NH_*Y^s and $H_*T_s(Z \otimes Y^\bullet)$ with T_s as in 10.8, we deduce 12.1(ii) for $D_r^{m,q}$. We let $i = k_*(i) \in N\pi_q(D_r^{m,q})^m$ and, using the triviality of $j_*(j) \in N\pi_{q+r-2}(D_{r-1}^{m,q})^{m+r} \approx 0$, we choose $j \in N\pi_{q+r-1}(D_r^{m,q})^{m+r}$ such that $D_r^{m,q}$ is a $d_r^{m,q}$-model. If $M^\bullet$ is a pointed cosimplicial space with $N\pi_{q+u-1}M^{m+u} \approx 0$ for $1 \leqq u \leqq r-1$, then for each $\alpha \in N\pi_q M^m$ there clearly exists $f: D_r^{m,q} \to M^\bullet$ in $\mathrm{Ho}(\nabla S_*)$ with $f_*(i) = \alpha$. *Thus for any $d_r^{m,q}$-model $M^\bullet$, there exists $f: D_r^{m,q} \to M^\bullet$ with $f_*(i) = i$, and $f: D_r^{m,q} \simeq M^\bullet$* by a homology argument. Moreover, by 12.2 $f_*(j) = j$ when $r < q$ and $f_*(j) - j \in [\pi_q M^{m+q}, \pi_q M^{m+q}]$ when $r = q$.

12.4. Pointed differentials in negative dimensions. For a pointed cosimplicial space $Y^\bullet$, $0 < q < m$, and $1 \leqq r \leqq q$, we define a relation $d_r: N\pi_q Y^m \rightharpoonup N\pi_{q+r-1}Y^{m+r}$ by letting $d^r(\alpha) = \beta$ whenever there is a map $u: D_r^{m,q} \to Y^\bullet$ in $\mathrm{Ho}(\nabla S_*)$ with $u_*(i) = \alpha$ and $u_*(j) = \beta$. This is clearly natural in $Y^\bullet$ and

LEMMA 12.5. *Suppose that the pointed differential d_r is defined on $N\pi_q(X^m, b)$, and let $f: Y^\bullet \to Z^\bullet$ be a pointed cosimplicial map such that $f_*: N\pi_t Y^s \approx N\pi_t Z^s$ for $m \leqq s \leqq m+r$, $q \leqq t \leqq q+r-1$, and $t - s \geqq -1$. Then the pointed differentials*

$$d_r: N\pi_q Y^m \rightharpoonup N\pi_{q+r-1}Y^{m+r}, \qquad d^r: N\pi_q Z^m \rightharpoonup N\pi_{q+r-1}Z^{m+r}$$

correspond under f_*.

PROOF. We may assume $Y^\bullet$ and $Z^\bullet$ fibrant and obtain equivalences

$$f: P_{r-h}\,\mathrm{Map}_*(D_k^{m,q}, Y^\bullet) \simeq P_{r-k}\,\mathrm{Map}_*(D_k^{m,q}, Z^\bullet)$$

for $1 \leqq k \leqq r$ by induction on k. The result follows when $k = r$.

For the fibrant cosimplicial space $X^\bullet$, we use the pointed differentials of 12.4 in the construction of 11.3 to give

12.6. General differentials in negative dimensions. These are relations

$$d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$$

for $0 < q < m$ and $1 \leqq r \leqq q$ where $b \in \mathrm{Tot}_r X^\bullet$ is a vertex such that the Whitehead products between $\pi_q(X^{m+r}, b)$ and the image of $b_*: \pi_r(Sk_r\Delta^{m+r}, 0) \to \pi_r(X^{m+r}, b)$ are trivial. This holds automatically when b is liftable to $\mathrm{Tot}_{r+1} X^\bullet$.

We also use 12.4 in 11.4 to give

12.7. Improved general differentials in negative dimensions. These are relations

$$d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$$

for $0 < q < m$ and $1 \leqq r \leqq q$ where $b \in \mathrm{Tot}_{r-1} X^\bullet$ is a vertex liftable to $\mathrm{Tot}_r X^\bullet$ and such that $[\pi_r(X^{m+r}, b), \pi_q(X^{m+r}, b)] = 0$.

The following results on these general differentials are easily verifed by reducing to the pointed case.

12.8. Formula for d_1. For $q < m$, a general $d_1: N\pi_q(X^m, b) \rightharpoonup N\pi_q(X^{m+1}, b)$ equals the function $(-1)^{q-m+1}\delta$ of 2.2.

12.9. Naturality properties. For $q < m$, a general $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ is natural in $X^\bullet$ and $b \in \mathrm{Tot}_r X^\bullet$ (or $b \in \mathrm{Tot}_{r-1} X^\bullet$ in the improved case 12.7), i.e. for cosimplicial maps and path classes.

12.10. Additivity properties. For $q < m$, a general $d_r: N\pi_q(X^m, q) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ always gives $d_r(0) = 0$ and is additive for $r < q$ (or $r = q$ in the improved case 12.7). This follows by constructing appropriate homotopy classes $D_r^{m,q} \to D_r^{m,q} \vee D_r^{m,q}$.

12.11. Domain and indeterminacy properties. For a general $d_r: N\pi_q(X^m, b) \rightharpoonup N\pi_{q+r-1}(X^{m+r}, b)$ with $q < m$ and $r \geqq 2$, the domain of d_r equals the kernel of d_{r-1}, and the indeterminacy of d_r equals the image of d_{r-1}. This follows since there are homotopy cofiberings

$$D_1^{m+r-1,q+r-2} \xrightarrow{j} D_{r-1}^{m,q} \to D_r^{m,q}, \qquad D_1^{m,q} \xrightarrow{i} D_r^{m,q} \to D_{r-1}^{m+1,q+1}$$

by 12.3 because the homotopy cofibre of i is a $d_{r-1}^{m+1,q+1}$-model.

12.12. Obstruction properties. For a vertex $b \in \mathrm{Tot}_{m-1} X^\bullet$ with $m \geqq 2$, the obstruction $c(b) \in N\pi_{m-1}(X^m, b)$ lies in the kernel of each general $d_r: N\pi_{m-1}(X^m, b) \rightharpoonup N\pi_{m+r-2}(X^{m+r}, b)$ with $r < m-1$ (or $r = m-1$ in the improved case 12.7). This follows using the universal example $Sk_{m-1}\Delta^\bullet$ made fibrant with $b = 1$, together with the following vanishing results obtained as in 12.2: (i) for $m \geqq 3$ and $t < 2m-3$, the group $N\pi_t Sk_{m-1}\Delta^s$ is isomorphic to $\pi_t S^{m-1}$ for $s = m$ and to 0 otherwise; (ii) for $m \geqq 3$ the group $N\pi_{2m-3} Sk_{m-1}\Delta^s / N[\pi_{m-1} Sk_{m-1}\Delta^s, \pi_{m-1} Sk_{m-1}\Delta^s]$ is isomorphic to $\pi_{2m-3} S^{m-1} / [\pi_{m-1} S^{m-1}, \pi_{m-1} S^{m-1}]$ when $s = m$ and to 0 otherwise; (iii) for $m = 2$, the group $N(Ab\pi_1 Sk_1 \Sigma^s)$ is isomorphic to Z for $s = 2$ and to 0 otherwise.

12.13. Composition properties. For $r, s \geqq 1$ and $1 \leqq q \leqq m$, let $b \in \mathrm{Tot}_{r+s-1} X^\bullet$ be a vertex with trivial Whitehead products between $\pi_q(X^{m+r+s}, b)$ and the image of

$$b_*: \pi_{r+s-1}(Sk_{r+s-1}\Delta^{m+r+s}, 0) \to \pi_{r+s-1}(X^{m+r+s}, b).$$

This holds automatically when b is liftable to $\mathrm{Tot}_{r+s} X^\bullet$. Assume one of the following:

(i) $r + s \leqq q$.
(ii) $r + s = q + 1$ and $[\pi_q(X^{m+r+s}, b), \pi_q(X^{m+r+s}, b)] = 0$.
(iii) $s < q = m$.
(iv) $s = q = m$ and $[\pi_q(X^{m+r+s}, b), \pi_{q+r-1}(X^{m+r+s}, b)] = 0$.

Then the general differentials

$$N\pi_q(X^m, b) \xrightarrow{d_r} N\pi_{q+r-1}(X^{m+r}, b) \xrightarrow{d_s} N\pi_{q+r+s-2}(X^{m+r+s}, b)$$

are defined and have range $d_r = \ker d_s$. After reducing to the pointed case, this follows for $q < m$ by applying 12.2 to the model $D_r^{m,q}$, and follows from $q = m$ by using 12.12.

12.14. Equivariance properties. For $q < m$, suppose that the general differentials

$$d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b), \qquad d_r: N\pi_1(X^0, b) \to N\pi_r(X^r, b)$$

respectively give $d_r y = z$ and $d_r w = 0$. Then the former gives $d_r(yw) = zw$ using the fundamental action of $\pi_1(X^0, b)$. This follows by the naturality of d_r in b since w lifts to $\pi_1(\mathrm{Tot}_r X^\bullet, b)$.

12.15. Hurewicz properties. For a general $d_r: N\pi_q(X^m, b) \to N\pi_{q+r-1}(X^{m+r}, b)$ with $q < m$ and ring R with identity, there is a corresponding homology differential $d_r: NH_q(X^m; R) \to NH_{q+r-1}(X^{m+r}; R)$ of [6] which gives $d_r(hx) = hy$ whenever $d_r x = y$ for $x \in N\pi_q(X^m, b)$. This follows by naturality since it holds for our model $D_r^{m,q}$ by definition.

§13. Construction of the homotopy spectral sequences

The spectral sequences of 2.4–2.6 are now immediately obtained by assembling the general differentials of Sections 11–12. We briefly outline the case of 2.4; the other cases are very similar but involve the improved general differentials. We shall use

13.1. Partial group actions. A *partial right-action* of a group G on a set W is a relation $*: W \times G \rightharpoonup W$ such that: (i) $w * e = w$ for each $w \in W$; (ii) $w * g = x$ implies $x * g^{-1} = w$; and (iii) $w * g = x$ and $x * h = y$ imply $w * gh = y$. There is an associated equivalence relation $\sim$ on W, where $w \sim x$ means $w * g = x$ for some $g \in G$, and $W/\sim$ is called the orbit set of W under the partial action. For an additive (but possibly non-abelian) group B with right-action by G, each crossed-additive relation $k: G \rightharpoonup B$ (as in 11.9) determines a partial right action of G on B where $b * g = bg + r$ whenever $k(g) = r$.

13.2. Construction of $\{E_r^{s,t}(X^\bullet, b)\}$ in 2.4. For $r \geqq 1$ and $b \in \mathrm{Tot}_{r-1} X^\bullet$ liftable to $\mathrm{Tot}_{2r-2} X^\bullet$, we construct $E_r^{s,t}(X^\bullet, b)$ as follows using the general differentials and their properties from Sections 11–12. When $r = 1$, $E_1^{s,t}(X^\bullet, b) = N\pi_t(X^\bullet, b)$. When $r \geqq 2$, $E_r^{0,0}(X^\bullet, b)$ is the free kernel of the bottom differential (11.14) $d_{r-1}: \pi_0 X^0 \rightharpoonup N\pi_{r-1}^{\mathrm{free}} X^r$ for $t \geqq 1$, $E_r^{0,t}(X^\bullet, b)$ is the kernel of the general $d_{r-1}: N\pi_t(X^0, b) \rightharpoonup N\pi_{t+r-2}(X^{r-1}, b)$ for $1 \leqq t \leqq r-1$; $E_r^{t,t}(X^\bullet, b)$ is the orbit set of the kernel of the general $d_{r-1}: N\pi_t(X^t, b) \rightharpoonup N\pi_{t+r-2}(X^{t+r-1}, b)$ under the partial right action associated via 13.1 with the crossed-additive (11.9) general $d_t: N\pi_1(X^0, b) \rightharpoonup N\pi_t(X^t, b)$; otherwise, $E_r^{s,t}(X^\bullet, b)$ is the quotient of the kernel of the general $d_{r-1}: N\pi_t(X^s, b) \rightharpoonup N\pi_{t+r-2}(X^{s+r-1}, b)$ by the image of the general $d_k: N\pi_{t-k+1}(X^{s-k}, b) \rightharpoonup N\pi_t(X^s, b)$ where $k = \min\{s, r-1\}$. The spectral sequence differentials are induced by the general differentials and the bottom differential.

APPENDIX

§14. On the homotopy theory of groupoids and cosimplicial groupoids

For a fibrant cosimplicial space $X^\bullet$, the cosimplicial fundamental groupoid $\pi_1^{gd} X^\bullet$ contains important information applicable to the lifting problem for vertices in $\{\mathrm{Tot}_s X^\bullet\}$. In 5.1 we obtained a natural correspondence

$$\pi^0 \pi_0 X^\bullet \approx (\pi_0 \mathrm{Tot}_0 X^\bullet)^{(1)} = \mathrm{Im}(\pi_0 \mathrm{Tot}_1 X^\bullet \to \pi_0 \mathrm{Tot}_0 X^\bullet).$$

Here, we shall introduce a set $\pi^1 \pi_1^{gd} X^\bullet$ and obtain a natural correspondence

$$\pi^1 \pi_1^{gd} X^\bullet \approx (\pi_0 \mathrm{Tot}_1 X^\bullet)^{(1)} = \mathrm{Im}(\pi_0 \mathrm{Tot}_2 X^\bullet \to \pi_0 \mathrm{Tot}_1 X^\bullet).$$

Thus a vertex $b \in X^0 = \mathrm{Tot}_0 X^\bullet$ will be liftable to $\mathrm{Tot}_2 X^\bullet$ iff $[b]$ belongs to $\pi^0 \pi_0 X^\bullet$ and lies in the image of the tower map $\pi^1 \pi_1^{gd} X^\bullet \to \pi^0 \pi_0 X^\bullet$. When the spaces X^s have abelian fundamental groups, the image condition can be replaced by the vanishing of the obstruction $\omega_2(b) \in \pi^2 \pi_1(X^\bullet, b)$ of 5.2, and such obstructions may possibly be defined in other settings using nonabelian cohomology. However, in general it seems homotopically most natural to use $\pi^1\pi_1^{gd}X^\bullet$. We start by assembling the homotopy theory of groupoids introduced in [1], [10], [14], and elsewhere. Then we prolong that theory to cosimplicial groupoids. For a cosimplicial groupoid $G^\bullet$, we introduce the total groupoid $\mathrm{Tot}\,G^\bullet$ and define $\pi^1G^\bullet = \pi_0 \mathrm{Tot}\,G^\bullet$ generalizing π^1 from the cosimplicial group case (2.2). This leads to the formula $\pi^1\pi_1^{gd}X^\bullet \approx (\pi_0 \mathrm{Tot}_1 X^\bullet)^{(1)}$.

14.1. The homotopy theory of groupoids. Let $\pi_1^{gd}: \mathrm{S} \to \mathrm{Gd}$ denote the fundamental groupoid functor from simplicial sets to groupoids. Thus for $K \in \mathrm{S}$, $\pi_1^{gd}K$ is the groupoid whose vertices (i.e. objects) are the vertices of K and whose morphisms from x to y are the path classes from y to x. The funtor π_1^{gd} has a right adjoint $B: \mathrm{Gd} \to \mathrm{S}$ where BG is the categorical nerve of G. The adjunction counit gives an isomorphism $\pi_1^{gd}BG \approx G$ for each $G \in \mathrm{Gd}$, and thus B is fully faithful. There is a mapping groupoid $\mathrm{Map}(G, H)$ whose vertices are the functors $G \to H$ and whose morphisms are the natural transformations; moreover,

$$\mathrm{Map}(F \times G, H) \approx \mathrm{Map}(F, \mathrm{Map}(G, H)) \qquad \text{for } F, G, H \in \mathrm{Gd}.$$

Since $\pi_1^{gd}: \mathrm{S} \to \mathrm{Gd}$ preserves finite products, adjunction gives

$$B\,\mathrm{Map}(\pi_1^{gd}K, H) \approx \mathrm{Map}(K, BH) \qquad \text{for } K \in S \quad \text{and} \quad H \in \mathrm{Gd}.$$

By [2], Gd is a closed model category [20], where a map $\phi: G \to H$ in Gd is: (i) a *weak equivalence* iff ϕ is a categorical equivalence; (ii) a *cofibration* iff ϕ is monic on vertices; and (iii) a *fibration* iff for each vertex $g \in G$ and morphism u to $\phi(g)$ in H there exists a morphism $\tilde{u}$ to g in G with $\phi(\tilde{u}) = u$. Moreover, Gd is a closed simplicial model category [20] with

$$G \otimes K = G \times \pi_1^{gd}K, \qquad G^K = \mathrm{Map}(\pi_1^{gd}K, G),$$

and function space $B\,\mathrm{Map}(G, H)$ for $G, H \in \mathrm{Gd}$ and $K \in S$. Examples of fibrations in Gd include full functors $\phi: G \to H$ surjective on vertices and *covering maps* $\phi: G \to H$, i.e. fibrations such that the above lifting $\tilde{u}$ exist uniquely. A map $\phi: G \to H$ in Gd is a weak equivalence or fibration iff $B\phi: BG \to BH$ is such. Whenever a map $f: K \to L$ in S is a weak equivalence, cofibration, or fibration, then $\pi_1^{gd}f: \pi_1^{gd}K \to \pi_1^{gd}L$ is such. Finally, note that each $G \in \mathrm{Gd}$ is cofibrant and fibrant.

For a groupoid G, let $\pi_0 G = \mathrm{Vert}\,G/\approx$. For a vertex $x \in G$, let $\pi_0(G, x) = (\pi_0 G, x)$, let $\pi_1(G, x) = \mathrm{Aut}\ x$, and let $\pi_i(G, x) = 0$ for $i \geqq 2$. Weak equivalences and fibrations in Gd produce the usual isomorphisms and exact sequences of π_*-terms. For $x \in G \in \mathrm{Gd}$ and $y \in Y \in \mathrm{S}$, there are natural isomorphisms $\pi_i(BG, x) \approx \pi_i(G, x)$ when $i \geqq 0$ and $\pi_j(Y, y) \approx \pi_j(\pi_1^{gd} Y, y)$ when $j = 0, 1$.

14.2. The homotopy theory of cosimplicial groupoids. The above π_1^{gd} and B prolong to adjoint functors $\pi_1^{gd} : \nabla\mathrm{S} \leftrightarrow \nabla\mathrm{Gd} : B$ between the categories of cosimplicial spaces and cosimplicial groupoids. There is a mapping groupoid $\mathrm{Map}(G^\bullet, H^\bullet)$ and are natural isomorphisms

$$\mathrm{Map}(F \times G^\bullet, H^\bullet) \approx \mathrm{Map}(F, \mathrm{Map}(G^\bullet, H^\bullet)) \approx \mathrm{Map}(G^\bullet, \mathrm{Map}(F, H^\bullet))$$

of groupoids for $F \in \mathrm{Gd}$ and $G^\bullet, H^\bullet \in \nabla\mathrm{Gd}$. There is also an adjunction isomorphism $B\,\mathrm{Map}(\pi_1^{gd} X^\bullet, H^\bullet) \approx \mathrm{Map}(X^\bullet, BH^\bullet)$ for $X^\bullet \in \nabla\mathrm{S}$ and $H^\bullet \in \nabla\mathrm{Gd}$.

The category $\nabla\mathrm{Gd}$ has a closed simplicial model category structure, similar to that of $\nabla\mathrm{S}$ [8, p. 277], where a map $\phi : G^\bullet \to H^\bullet$ in $\nabla\mathrm{Gd}$ is: (i) a *weak equivalence* iff $\phi : G^m \to H^m$ is a weak equivalence in Gd for $m \geqq 0$; (ii) a *cofibration* iff ϕ restricts to a cofibration of the vertex cosimplicial sets; a *fibration* iff the maps

$$(\phi, s) : G^m \to H^m \times_{M^{m-1}H^\bullet} M^{m-1} G^\bullet$$

are fibrations in Gd for $m \geqq 0$ where (ϕ, s) is as in [8, p. 275]. A map $\phi : G^\bullet \to H^\bullet$ in $\nabla\mathrm{Gd}$ is a weak equivalence or fibration iff $B\phi : BG^\bullet \to BH^\bullet$ is such. Whenever a map $f : X^\bullet \to Y^\bullet$ in $\nabla\mathrm{S}$ is a weak equivalence or cofibration, then $\pi_1^{gd} f : \pi_1^{gd} X^\bullet \to \pi_1^{gd} Y^\bullet$ is such; likewise, whenever $f : X^\bullet \to Y^\bullet$ is a fibration of termwise connected fibrant objects, then $\pi_1^{gd} f$ is such by [6, 5.2–5.3] since $B\pi_1^{gd}$ is a generalized Postnikov functor. If $\phi : G^\bullet \to H^\bullet$ is a map of termwise connected cosimplicial groupoids such that each $\phi : G^m \to H^m$ is full and each $(\phi, s) : G^m \to H^m \times_{M^{m-1}H} M^{m-1}G$ is surjective on vertices, then each (ϕ, s) is full and $\phi : G^\bullet \to H^\bullet$ is a fibration as in [6, §5]. In particular, a quotient homomorphism of cosimplicial groups is a fibration in $\nabla\mathrm{Gd}$. However, $\nabla\mathrm{Gd}$ has non-fibrants such as $\pi_1^{gd} \Delta^\bullet$.

14.3. Total groupoids. A cosimplicial groupoid $G^\bullet$ has a *total groupoid* $\mathrm{Tot}\, G^\bullet = \mathrm{Map}(\pi_1^{gd} \Delta^\bullet, G^\bullet)$ and tower of groupoids $\mathrm{Tot}_m G^\bullet = \mathrm{Map}(\pi_1^{gd} Sk_m \Delta^\bullet, G^\bullet)$, satisfying $B\,\mathrm{Tot}\, G^\bullet \approx \mathrm{Tot}\, BG^\bullet$ and $B\,\mathrm{Tot}_m G^\bullet \approx \mathrm{Tot}_m BG^\bullet$ by adjunction. In more detail, $\mathrm{Tot}_0 G^\bullet = G^0$; $\mathrm{Tot}_1 G^\bullet$ has vertices (x, u) where $x \in \mathrm{obj}\, G^0$ and $u : d^0 x \to d^1 x$ in G^1 with $s^0 u = 1$, and has morphisms $\alpha : (x, u) \to (y, v)$ for $\alpha : x \to y$ in G^0 with $(d^1\alpha)u = v(d^0\alpha)$; the tower map $\mathrm{Tot}_1 G^\bullet \to \mathrm{Tot}_0 G^\bullet$ is a covering sending each (x, u) to x; $\mathrm{Tot}_2 G^\bullet$ is the full subgroupoid of $\mathrm{Tot}_1 G^\bullet$ given by all (x, u) with $(d^2u)(d^0u)(d^1u)^{-1} = 1$; and $\mathrm{Tot}\, G^\bullet = \mathrm{Tot}_m G^\bullet = \mathrm{Tot}_2 G^\bullet$ for $m \geqq 2$. For a map $\phi : G^\bullet \to H^\bullet$ in ∇Gd: (i) if ϕ is a weak equivalence (or more generally if $\phi : G^n \to H^n$ is an equivalence for $n = 0, 1$ and faithful for $n = 2$), then each $\mathrm{Tot}_m \phi : \mathrm{Tot}_m G^\bullet \to \mathrm{Tot}_m H^\bullet$ is an equivalence; and (ii) if ϕ is a fibration (or more generally if $\phi : G^0 \to H^0$ is a fibration), then each $\mathrm{Tot}_m \phi : \mathrm{Tot}_m G^\bullet \to \mathrm{Tot}_m H^\bullet$ is a fibration.

For a cosimplicial groupoid $G^\bullet$, we define $\pi^1 G^\bullet = \pi_0 \mathrm{Tot}\, G^\bullet$ and $\pi^0(G^\bullet, b) = \pi_1(\mathrm{Tot}\, G^\bullet, b)$ for any vertex $b \in \mathrm{Tot}\, G^\bullet$. There are evident π^* isomorphisms for weak equivalences and π^* exact sequences for fibrations of cosimplicial groupoids. When $G^\bullet$ is termwise connected with a vertex $b \in \mathrm{Tot}\, G^\bullet$, the definitions of 2.2 apply to the cosimplicial group $\pi_1(G^\bullet, b)$, and there is a bijection $\pi^1\pi_1(G^\bullet, b) \approx \pi^1 G^\bullet$ and isomorphisms $\pi^0\pi_1(G^\bullet, b) \approx \pi^0(G^\bullet, b)$ obtained using the weak equivalences

$$\pi_1(G^\bullet, b) \leftarrow \bar{G}^\bullet \xrightarrow{\beta} G^\bullet$$

where β is given by $b : \pi_1^{gd}\Delta^\bullet \to G^\bullet$ on vertices. Finally

PROPOSITION 14.4. *For a fibrant cosimplicial space $X^\bullet$, there is a natural bijection* $(\pi_0 \mathrm{Tot}_1 X^\bullet)^{(1)} \approx \pi^1\pi_1^{gd}X^\bullet$

PROOF. It suffices to show that $X^\bullet \to B\pi_1^{gd}X^\bullet$ induces

$$(\pi_0 \mathrm{Tot}_1 X^\bullet)^{(1)} \approx (\pi_0 \mathrm{Tot}_1 B\pi_1^{gd}X^\bullet)^{(1)}.$$

since clearly

$$(\pi_0 \mathrm{Tot}_1 B\pi_1^{gd}X^\bullet)^{(1)} \approx (\pi_0 \mathrm{Tot}_1 \pi_1^{gd}X^\bullet)^{(1)} \approx \pi^1\pi_1^{gd}X^\bullet.$$

Assuming by reduction to components that $X^\bullet$ is connected, this follows by 3.1 and 5.1 since $X^\bullet \to B\pi_1^{gd}X^\bullet$ is a map of fibrant objects inducing $\pi_1^{gd}(X^\bullet) \approx \pi_1^{gd}(B\pi_1^{gd}X^\bullet)$.

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